ON A RAMIFICATION BOUND OF TORSION SEMI-STABLE REPRESENTATIONS OVER A LOCAL FIELD

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Abstract. Let p be a rational prime, k be a perfect field of characteristic p, $W = W(k)$ be the ring of Witt vectors, K be a finite totally ramified extension of $\mathrm{Frac}(W)$ of degree e and r be a non-negative integer satisfying $r < p-1$. In this paper, we prove the upper numbering ramification group $G_K^{(j)}$ for $j > u(K,r,n)$ acts trivially on the p^n-torsion semi-stable G_K-representations with Hodge-Tate weights in $\{0,\ldots,r\}$, where $u(K,0,n) = 0$, $u(K,1,n) = 1 + e(n + 1/(p-1))$ and $u(K,r,n) = 1 - p^{-n} + e(n + r/(p-1))$ for $1 < r < p-1$.

1. Introduction

Let p be a rational prime, k be a perfect field of characteristic p, $W = W(k)$ be the ring of Witt vectors and K be a finite totally ramified extension of $K_0 = \mathrm{Frac}(W)$ of degree $e = e(K)$. We normalize the valuation v_K of K as $v_K(p) = e$ and extend this to any algebraic closure of K. Let the maximal ideal of K be denoted by m_K, an algebraic closure of K by $\bar{K}$ and the absolute Galois group of K by $G_K = \mathrm{Gal}(\bar{K}/K)$. Let $G_K^{(j)}$ denote the j-th upper numbering ramification group in the sense of [10]. Namely, we put $G_K^{(j)} = G_K^{j-1}$, where the latter is the upper numbering ramification group defined in [18].

Consider a proper smooth scheme X_K over K and put $X_{\bar{K}} = X_K \times_K \bar{K}$. Let $\mathcal{L} \supseteq \mathcal{L}'$ be G_K-stable $\mathbb{Z}_p$-lattices in the r-th etale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ such that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by p^n. In [10], Fontaine conjectured the upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K-modules $\mathcal{L}/\mathcal{L}'$ and $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ for $j > e(n + r/(p-1))$ if X_K has good reduction. For $e = 1$ and $r < p-1$, this conjecture was proved independently by himself ([11], for $n = 1$) and Abrashkin ([3], for any n), using the theory of Fontaine-Laffaille ([13]) and the comparison theorem of Fontaine-Messing ([14], see also [5] and [7]) between the etale cohomology groups of X_K and the crystalline cohomology groups of the reduction of X_K. From these results, they also showed some rareness of a proper smooth

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scheme over $\mathbb{Q}$ with everywhere good reduction ([11, Théorème 1], [2, Section 7]). In fact, they proved this ramification bound for the torsion crystalline representations of G_K with Hodge-Tate weights in $\{0,\ldots,r\}$ in the case where K is absolutely unramified.

On the other hand, for a torsion semi-stable representation with Hodge-Tate weights in the same range, a similar ramification bound for $e=1$ and $n=1$ is obtained by Breuil (see [7, Proposition 9.2.2.2]). He showed, assuming the Griffiths transversality which in general does not hold, that if $e=1$ and $r<p-1$, then the ramification group $G_K^{(j)}$ acts trivially on the mod p semi-stable representations for $j>2+1/(p-1)$.

In this paper, we prove a ramification bound for the torsion semi-stable representations of G_K with Hodge-Tate weights in $\{0,\ldots,r\}$ with no assumption on e but under the assumption $r<p-1$. Let π be a uniformizer of K, $E(u)\in W[u]$ be the Eisenstein polynomial of π over W and S be the p-adic completion of the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$. Consider a category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ of filtered (ϕ_r,N)-modules over the ring S and a G_K-module

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})=\mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M},\hat{A}_{\mathrm{st},\infty})$$

for $\mathcal{M}\in\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$, where $\hat{A}_{\mathrm{st},\infty}$ is a p-adic period ring ([6]). Then our main theorem is the following.

Theorem 1.1. *Let r be a non-negative integer such that $r<p-1$. Let $\mathcal{M}$ be an object of the category $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ which is killed by p^n. Then the j-th upper numbering ramification group $G_K^{(j)}$ acts trivially on the G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})$ for $j>u(K,r,n)$, where*

$$u(K,r,n)=\begin{cases}0 & (r=0),\\ 1+e(n+\frac{1}{p-1}) & (r=1),\\ 1-\frac{1}{p^n}+e(n+\frac{r}{p-1}) & (1<r<p-1).\end{cases}$$

We can check that this bound is sharp for $r\leq 1$ (Remark 5.15).

From this theorem and [10, Proposition 1.3], we have the following corollary.

Corollary 1.2. *Let the notation be as in the theorem and L be the finite extension of K cut out by the G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})$. Namely, the finite extension L is defined by*

$$G_L=\mathrm{Ker}(G_K\to\mathrm{Aut}(T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}))).$$

Let $\mathfrak{D}_{L/K}$ denote the different of the extension L/K. Then we have the inequality

$$v_K(\mathfrak{D}_{L/K})<u(K,r,n)$$

for $r>0$ and $v_K(\mathfrak{D}_{L/K})=0$ for $r=0$.

Combining these results with a theorem of Liu ([17, Theorem 2.3.5]) or a theorem of Caruso ([8, Théorème 1.1]), we will show the corollary below.

Corollary 1.3. *Let r be a non-negative integer such that $r < p-1$. Then the same bounds as in Theorem 1.1 and Corollary 1.2 are also valid for the torsion G_K-modules of the following two cases:*

(1) the G_K-module $\mathcal{L}/\mathcal{L}'$, where $\mathcal{L} \supseteq \mathcal{L}'$ are G_K-stable $\mathbb{Z}_p$-lattices in a semi-stable p-adic representation V with Hodge-Tate weights in $\{0,\dots,r\}$ such that $\mathcal{L}/\mathcal{L}'$ is killed by p^n.

(2) the G_K-module $H^r_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$, where X_K is a proper smooth algebraic variety over K which has a proper semi-stable model over $\mathcal{O}_K$ and r satisfies $er < p-1$ for $n=1$ and $e(r+1) < p-1$ for $n > 1$.

For the proof of Theorem 1.1, we basically follow a beautiful argument of Abrashkin ([3]). We may assume $p \geq 3$ and $r \geq 1$. Consider the finite Galois extension

$$F_n = K(\pi^{1/p^n}, \zeta_{p^{n+1}})$$

of K whose upper ramification is bounded by $u(K,r,n)$. Let L_n be the finite Galois extension of F_n cut out by $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})|_{G_{F_n}}$. Then we bound the ramification of L_n over K. For this, we show that to study this G_{F_n}-module we can use a variant over a smaller coefficient ring Σ of filtered (ϕ_r, N)-modules over S. In precise, we set

$$\Sigma = W[[u, E(u)^p/p]].$$

This ring Σ is small enough for the method of Abrashkin, in which he uses filtered modules of Fontaine-Laffaille ([13]) whose coefficient ring is W, to work also in the case where K is absolutely ramified.

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2. Filtered (ϕ_r, N)-modules of Breuil

In this section, we recall the theory of filtered (ϕ_r, N)-modules over S of Breuil, which is developed by himself and most recently by Caruso and Liu (see for example [6], [8], [17], [9]). In what follows, we always take the divided power envelope of a W-algebra with the compatibility condition with the natural divided power structure on pW.

Let p be a rational prime and σ be the Frobenius endomorphism of W. We fix once and for all a uniformizer π of K and a system $\{\pi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of p-power roots of π in $\bar{K}$ such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$ for any n. Let $E(u)$

be the Eisenstein polynomial of π over W and set $S=(W[u]^{\mathrm{PD}})^{\wedge}$, where PD means the divided power envelope and this is taken with respect to the ideal $(E(u))$, and $\wedge$ means the p-adic completion. The ring S is endowed with the σ-semilinear endomorphism $\phi: u\mapsto u^p$ and a natural filtration $\mathrm{Fil}^t S$ induced by the divided power structure such that $\phi(\mathrm{Fil}^t S)\subseteq p^t S$ for $0\leq t\leq p-1$. We set $\phi_t=p^{-t}\phi|_{\mathrm{Fil}^t S}$ and $c=\phi_1(E(u))\in S^\times$. Let N denote the W-linear derivation on S defined by the formula $N(u)=-u$. We also define a filtration, ϕ, ϕ_t and N on $S_n=S/p^nS$ similarly.

Let $r\in\{0,\ldots,p-2\}$ be an integer. Set $'\mathrm{Mod}^{r,\phi,N}_{/S}$ to be the category consisting of the following data:

- an S-module $\mathcal{M}$ and its S-submodule $\mathrm{Fil}^r\mathcal{M}$ containing $\mathrm{Fil}^r S\cdot\mathcal{M}$,
- a ϕ-semilinear map $\phi_r:\mathrm{Fil}^r\mathcal{M}\to\mathcal{M}$ satisfying

$$\phi_r(s_r m)=\phi_r(s_r)\phi(m)$$

  for any $s_r\in\mathrm{Fil}^r S$ and $m\in\mathcal{M}$, where we set $\phi(m)=c^{-r}\phi_r(E(u)^r m)$,
- a W-linear map $N:\mathcal{M}\to\mathcal{M}$ such that
  - $N(sm)=N(s)m+sN(m)$ for any $s\in S$ and $m\in\mathcal{M}$,
  - $E(u)N(\mathrm{Fil}^r\mathcal{M})\subseteq\mathrm{Fil}^r\mathcal{M}$,
  - the following diagram is commutative:

$$\begin{array}{ccc}\mathrm{Fil}^r\mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M}\\ {\scriptstyle E(u)N}\downarrow & & \downarrow{\scriptstyle cN}\\ \mathrm{Fil}^r\mathcal{M} & \xrightarrow[\phi_r]{} & \mathcal{M},\end{array}$$

and the morphisms of $'\mathrm{Mod}^{r,\phi,N}_{/S}$ are defined to be the S-linear maps preserving Fil^r and commuting with ϕ_r and N. The category defined in the same way but dropping the data N is denoted by $'\mathrm{Mod}^{r,\phi}_{/S}$. These categories have obvious notions of exact sequences. Let $\mathrm{Mod}^{r,\phi,N}_{/S_1}$ denote the full subcategory of $'\mathrm{Mod}^{r,\phi,N}_{/S}$ consisting of $\mathcal{M}$ such that $\mathcal{M}$ is free of finite rank over S_1 and generated as an S_1-module by the image of ϕ_r. We write $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ for the smallest full subcategory which contains $\mathrm{Mod}^{r,\phi,N}_{/S_1}$ and is stable under extensions. We let $\mathrm{Mod}^{r,\phi,N}_{/S}$ denote the full subcategory consisting of $\mathcal{M}$ such that

- the S-module $\mathcal{M}$ is free of finite rank and generated by the image of ϕ_r,
- the quotient $\mathcal{M}/\mathrm{Fil}^r\mathcal{M}$ is p-torsion free.

We define full subcategories $\mathrm{Mod}^{r,\phi}_{/S_1}$, $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ and $\mathrm{Mod}^{r,\phi}_{/S}$ of $'\mathrm{Mod}^{r,\phi}_{/S}$ in a similar way. For $\hat{\mathcal{M}}\in\mathrm{Mod}^{r,\phi,N}_{/S}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/S}$), the quotient $\hat{\mathcal{M}}/p^n\hat{\mathcal{M}}$ has a natural structure as an object of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/S_\infty}$).

For p-torsion objects, we also have the following categories. Consider the k-algebra $k[u]/(u^{ep}) \cong S_1/\mathrm{Fil}^p S_1$ and let this algebra be denoted by $\tilde{S}_1$. The algebra $\tilde{S}_1$ is equipped with the natural filtration, ϕ and N induced by those of S. Namely, $\mathrm{Fil}^t\tilde{S}_1 = u^{et}\tilde{S}_1$, $\phi(u) = u^p$ and $N(u) = -u$. Let $'\mathrm{Mod}^{r,\phi,N}_{/\tilde{S}_1}$ denote the category consisting of the following data:

- an $\tilde{S}_1$-module $\tilde{\mathcal{M}}$ and its $\tilde{S}_1$-submodule $\mathrm{Fil}^r\tilde{\mathcal{M}}$ containing $u^{er}\tilde{\mathcal{M}}$,
- a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$,
- a k-linear map $N : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ such that
  - $N(sm) = N(s)m + sN(m)$ for any $s \in \tilde{S}_1$ and $m \in \tilde{\mathcal{M}}$,
  - $u^e N(\mathrm{Fil}^r\tilde{\mathcal{M}}) \subseteq \mathrm{Fil}^r\tilde{\mathcal{M}}$,
  - the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Fil}^r\tilde{\mathcal{M}} \arrow[r, "\phi_r"] \arrow[d, "u^e N"'] & \tilde{\mathcal{M}} \arrow[d, "cN"] \\ \mathrm{Fil}^r\tilde{\mathcal{M}} \arrow[r, "\phi_r"'] & \tilde{\mathcal{M}}, \end{tikzcd}$$

and whose morphisms are defined as before. Its full subcategory $\mathrm{Mod}^{r,\phi,N}_{/\tilde{S}_1}$ is defined by the following condition:

- As an $\tilde{S}_1$-module, $\tilde{\mathcal{M}}$ is free of finite rank and generated by the image of ϕ_r.

We define categories $'\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ and $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ similarly. Then we can show as in the proof of [4, Proposition 2.2.2.1] that the natural functor $\mathcal{M} \mapsto \mathcal{M}/\mathrm{Fil}^p S \cdot \mathcal{M}$ induces equivalences of categories $T : \mathrm{Mod}^{r,\phi,N}_{/S_1} \to \mathrm{Mod}^{r,\phi,N}_{/\tilde{S}_1}$ and $T_0 : \mathrm{Mod}^{r,\phi}_{/S_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$.

For $r = 0$, let $\mathrm{Mod}^{\phi}_{/W_\infty}$ be the category consisting of the following data:

- a finite torsion W-module $\tilde{M}$,
- a σ-semilinear automorphism $\phi : \tilde{M} \to \tilde{M}$.

Let κ be the kernel of the natural surjection $S \to W$ defined by $u \mapsto 0$. Since $\mathrm{Tor}_1^S(\mathcal{M}, S/\kappa S) = 0$ for any $\mathcal{M} \in \mathrm{Mod}^{0,\phi}_{/S_\infty}$, the proofs of [8, Lemme 2.2.7, Proposition 2.2.8] work also for the category $\mathrm{Mod}^{0,\phi,N}_{/S_\infty}$ and we have a commutative diagram of categories

$$\begin{tikzcd} \mathrm{Mod}^{0,\phi,N}_{/S_\infty} \arrow[r] \arrow[dr] & \mathrm{Mod}^{0,\phi}_{/S_\infty} \arrow[d] \\ & \mathrm{Mod}^{\phi}_{/W_\infty}, \end{tikzcd}$$

where the downward arrows and horizontal arrow are defined by $\mathcal{M} \mapsto \mathcal{M}/\kappa\mathcal{M}$ and forgetting N respectively and these three arrows are equivalences of categories.

Let A_{crys} and $\hat{A}_{\mathrm{st}}$ be p-adic period rings. These are constructed as follows. Put $\tilde{\mathcal{O}}_{\bar{K}} = \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. Set R to be the ring

$$R = \varprojlim(\tilde{\mathcal{O}}_{\bar{K}} \leftarrow \tilde{\mathcal{O}}_{\bar{K}} \leftarrow \cdots),$$

where every arrow is the p-power map. For an element $x = (x_i)_{i\in\mathbb{Z}_{\geq 0}} \in R$ and an integer $n \geq 0$, we set

$$x^{(n)} = \lim_{m\to\infty} \hat{x}_{n+m}^{p^m} \in \mathcal{O}_{\mathbb{C}},$$

where $\hat{x}_i$ is a lift of x_i in $\mathcal{O}_{\bar{K}}$ and $\mathcal{O}_{\mathbb{C}}$ is the p-adic completion of $\mathcal{O}_{\bar{K}}$. Let v_p denote the valuation of $\mathcal{O}_{\mathbb{C}}$ normalized as $v_p(p) = 1$. Then the ring R is a complete valuation ring whose valuation of an element $x \in R$ is given by $v_R(x) = v_p(x^{(0)})$. We define a natural ring homomorphism θ by

$$\begin{aligned} \theta : W(R) &\to \mathcal{O}_{\mathbb{C}} \\ (x_0, x_1, \ldots) &\mapsto \sum_{n\geq 0} p^n x_n^{(n)}. \end{aligned}$$

Then A_{crys} is the p-adic completion of the divided power envelope of $W(R)$ with respect to the principal ideal $\mathrm{Ker}(\theta)$ and $\hat{A}_{\mathrm{st}}$ is the p-adic completion of the divided power polynomial ring $A_{\mathrm{crys}}\langle X\rangle$ over A_{crys}. We set $A_{\mathrm{crys},\infty} = A_{\mathrm{crys}} \otimes_W K_0/W$ and $\hat{A}_{\mathrm{st},\infty} = \hat{A}_{\mathrm{st}} \otimes_W K_0/W$. Put $\underline{\pi} = (\pi_n)_{n\in\mathbb{Z}_{\geq 0}} \in R$, where we abusively let π_n denote the image of $\pi_n \in \mathcal{O}_{\bar{K}}$ in $\tilde{\mathcal{O}}_{\bar{K}}$. These rings are considered as S-algebras by the ring homomorphisms $S \to \hat{A}_{\mathrm{st}}$ and $\hat{A}_{\mathrm{st}} \to A_{\mathrm{crys}}$ which are defined by $u \mapsto [\underline{\pi}]/(1+X)$ and $X \mapsto 0$, respectively. The ring A_{crys} is endowed with a natural filtration induced by the divided power structure, a natural Frobenius endomorphism ϕ and the ϕ-semilinear map $\phi_t = p^{-t}\phi|_{\mathrm{Fil}^t A_{\mathrm{crys}}}$. With these structures, A_{crys} and $A_{\mathrm{crys},\infty}$ are considered as objects of ${}'\mathrm{Mod}^{r,\phi}_{/S}$. Moreover, the absolute Galois group G_K acts naturally on these two rings. As for $\hat{A}_{\mathrm{st}}$, its filtration is defined by

$$\mathrm{Fil}^t \hat{A}_{\mathrm{st}} = \left\{ \sum_{i\geq 0} a_i \frac{X^i}{i!} \;\middle|\; a_i \in \mathrm{Fil}^{t-i} A_{\mathrm{crys}},\ \lim_{i\to\infty} a_i = 0 \right\}$$

and the Frobenius structure of A_{crys} extends to $\hat{A}_{\mathrm{st}}$ by

$$\begin{aligned} \phi(X) &= (1+X)^p - 1, \\ \phi_t &= p^{-t}\phi|_{\mathrm{Fil}^t \hat{A}_{\mathrm{st}}}. \end{aligned}$$

We write N also for the A_{crys}-linear derivation on $\hat{A}_{\mathrm{st}}$ defined by $N(X) = 1+X$. The rings $\hat{A}_{\mathrm{st}}$ and $\hat{A}_{\mathrm{st},\infty}$ are objects of ${}'\mathrm{Mod}^{r,\phi,N}_{/S}$. The G_K-action on A_{crys} naturally extends to an action on $\hat{A}_{\mathrm{st}}$. Indeed, the action of $g \in G_K$ on $\hat{A}_{\mathrm{st}}$ is defined by the formula

$$g(X) = [\underline{\varepsilon}(g)](1+X) - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\underline{\varepsilon}(g) = (\varepsilon_n(g))_{n\in\mathbb{Z}_{\geq 0}} \in R$ with the abusive notation as above.

These rings have other descriptions, as follows. For an integer $n \geq 1$, put $W_n = W/p^nW$ and let $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ be the ring of Witt vectors of length n associated to $\tilde{\mathcal{O}}_{\bar{K}}$. We define a W_n-algebra structure on $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ by twisting the natural W_n-algebra structure by σ^{-n}. Then the natural ring homomorphism

$$\begin{aligned}\theta_n : W_n(\tilde{\mathcal{O}}_{\bar{K}}) &\to \mathcal{O}_{\bar{K}}/p^n\mathcal{O}_{\bar{K}}\\ (a_0,\ldots,a_{n-1}) &\mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}},\end{aligned}$$

where $\hat{a}_i$ is a lift of a_i in $\mathcal{O}_{\bar{K}}$, is W_n-linear. Let us denote $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$ the divided power envelope of $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ with respect to the ideal $\mathrm{Ker}(\theta_n)$. This ring is considered as an S-algebra by $u \mapsto [\pi_n]$. This ring also has a natural filtration defined by the divided power structure, and a natural G_K-module structure. The Frobenius endomorphism of the ring of Witt vectors induces on this ring a ϕ-semilinear Frobenius endomorphism, which is denoted also by ϕ. Then, by the S-linear transition maps

$$\begin{aligned}W_{n+1}^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}) &\to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\\ (a_0,\ldots,a_n) &\mapsto (a_0^p,\ldots,a_{n-1}^p),\end{aligned}$$

these S-algebras form a projective system compatible with all the structures. Using this transition map, a ϕ-semilinear map

$$\phi_r : \mathrm{Fil}^r W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}) \to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$$

is defined by setting $\phi_r(x)$ to be the image of $p^{-r}\phi(\hat{x})$, where $\hat{x}$ is a lift of x in $\mathrm{Fil}^r W_{n+r}^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$. By definition, the maps ϕ_r are also compatible with the transition maps. The S-algebra $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$ is considered as an object of $'\mathrm{Mod}_{/S}^{r,\phi}$. Then we have a natural isomorphism in $'\mathrm{Mod}_{/S}^{r,\phi}$

$$\begin{aligned}A_{\mathrm{crys}}/p^nA_{\mathrm{crys}} &\to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\\ (x_0,\ldots,x_{n-1}) &\mapsto (x_{0,n},\ldots,x_{n-1,n}),\end{aligned}$$

where we set $x_i = (x_{i,k})_{k\in\mathbb{Z}_{\geq 0}}$ for $x_i \in R$.

Similarly, the divided power polynomial ring $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle$ over $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$ is considered as an S-algebra by $u \mapsto [\pi_n]/(1+X)$. This ring has a natural filtration coming from the divided power structure. We define a G_K-action on this ring by

$$g(X) = [\varepsilon_n(g)](1+X) - 1.$$

We also define a ϕ-semilinear Frobenius endomorphism, which we also write as ϕ, by $\phi(X) = (1+X)^p - 1$ and a $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$-linear derivation N by $N(X) = 1 + X$. These rings form a projective system of S-algebras compatible with

all the structures by the transition maps defined by the maps above and $X \mapsto X$. We define ϕ-semilinear maps

$$\phi_r : \mathrm{Fil}^r W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle \to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle$$

compatible with the transition maps as before. The S-algebra $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle$ is considered as an object of $'\mathrm{Mod}^{r,\phi,N}_{/S}$ and there exists a natural isomorphism in $'\mathrm{Mod}^{r,\phi,N}_{/S}$

$$\begin{aligned} \hat{A}_{\mathrm{st}}/p^n\hat{A}_{\mathrm{st}} &\to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle \\ (x_0,\dots,x_{n-1}) &\mapsto (x_{0,n},\dots,x_{n-1,n}) \\ X &\mapsto X \end{aligned}$$

which is G_K-linear.

Put $K_n = K(\pi_n)$ and $K_\infty = \cup_n K_n$. For $\mathcal{M} \in \mathrm{Mod}^{r,\phi,N}_{/S_\infty}$, we define a G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})$ to be

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \hat{A}_{\mathrm{st},\infty}).$$

When $\mathcal{M}$ is killed by p^n, we have a natural identification of G_K-modules

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle).$$

Note that the G_K-module on the right-hand side is independent of the choice of π_k for $k > n$. Since the natural map

$$\begin{aligned} W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})\langle X\rangle &\to W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}) \\ X &\mapsto 0 \end{aligned}$$

is G_{K_n}-linear, we also have a G_{K_n}-linear isomorphism ([6, Lemme 2.3.1.1])

$$T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})|_{G_{K_n}} \to \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})).$$

On the other hand, for $r = 0$, the proof of [8, Proposition 2.3.13] shows that the G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})$ is unramified for any $\mathcal{M} \in \mathrm{Mod}^{0,\phi,N}_{/S_\infty}$.

A variant of filtered (ϕ_r, N)-modules over S is also introduced by Breuil and Kisin, and developed also by Caruso and Liu (see for example [15], [16], [17], [9]). Put $\mathfrak{S} = W[[u]]$ and let $\phi : \mathfrak{S} \to \mathfrak{S}$ be the σ-semilinear Frobenius endomorphism defined by $\phi(u) = u^p$. Let $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ denote the category consisting of the following data:

- an $\mathfrak{S}$-module $\mathfrak{M}$,
- a ϕ-semilinear map $\mathfrak{M} \to \mathfrak{M}$, which is denoted also by ϕ, such that the cokernel of the map $1 \otimes \phi : \phi^*\mathfrak{M} \to \mathfrak{M}$, where we set $\phi^*\mathfrak{M} = \mathfrak{S} \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$, is killed by $E(u)^r$,

and whose morphisms are defined as before. The full subcategory of $'\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$ consisting of $\mathfrak{M}$ such that $\mathfrak{M}$ is free of finite rank over $\mathfrak{S}/p\mathfrak{S}$ (*resp.* over $\mathfrak{S}$) is denoted by $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}}$). We let $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$ denote the smallest full subcategory which contains $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and is stable under extensions, as

before. Then we have an exact functor ([9, Proposition 2.1.2], see also [15, Proposition 1.1.11])

$$\mathcal{M}_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}.$$

For $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, the filtered ϕ_r-module $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$ over S is defined as follows:

- $\mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$,
- $\mathrm{Fil}^r\mathcal{M} = \mathrm{Ker}(\mathcal{M} \overset{1\otimes\phi}{\to} S \otimes_{\mathfrak{S}} \mathfrak{M} \to (S/\mathrm{Fil}^r S) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \mathrm{Fil}^r\mathcal{M} \overset{1\otimes\phi}{\to} \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_r\otimes 1}{\to} S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}$.

We write $\mathcal{M}_{\mathfrak{S}}$ for the functor $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}} \to \mathrm{Mod}^{r,\phi}_{/S}$ defined similarly.

3. Filtered ϕ_r-modules over Σ

In this section, we define another variant $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ of the category $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ over a subring Σ of the ring S, and prove that they are categorically equivalent.

Let p be a rational prime and r be an integer such that $0 \le r < p-1$. Consider the W-algebra $\Sigma = W[[u,Y]]/(E(u)^p - pY)$ as in [6, Subsection 3.2]. We regard Σ as a subring of S by the map sending Y to $E(u)^p/p$. Then the element $c = \phi_1(E(u)) \in S^\times$ is contained in $\Sigma^\times$. We define on Σ a σ-semilinear Frobenius endomorphism ϕ by $\phi(u) = u^p$ and $\phi(Y) = p^{p-1}c^p$. Put $\mathrm{Fil}^t\Sigma = (E(u)^t, Y)$ for $0 \le t \le p-1$ and $\mathrm{Fil}^p\Sigma = (Y)$. Then we have $\phi(\mathrm{Fil}^t\Sigma) \subseteq p^t\Sigma$ for $0 \le t \le p-1$. We put $\phi_t = p^{-t}\phi|_{\mathrm{Fil}^t\Sigma}$. We also set $\Sigma_n = \Sigma/p^n\Sigma$ and put on this ring the natural structures induced by those of Σ.

We define a category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ of filtered ϕ_r-modules over Σ to be the category consisting of the following data:

- a Σ-module M and its Σ-submodule $\mathrm{Fil}^r M$ containing $\mathrm{Fil}^r\Sigma \cdot M$,
- a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r M \to M$ satisfying $\phi_r(s_r m) = \phi_r(s_r)\phi(m)$ for any $s_r \in \mathrm{Fil}^r\Sigma$ and $m \in M$, where we set $\phi(m) = c^{-r}\phi_r(E(u)^r m)$,

and whose morphisms are defined in the same manner as ${}'\mathrm{Mod}^{r,\phi}_{/S}$. This category has a natural notion of exact sequences. We define its full subcategory $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ to be the category consisting of M which is free of finite rank and generated by the image of ϕ_r as a Σ_1-module. We also let $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ denote the smallest full subcategory of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ which contains $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ and is stable under extensions. Moreover, we define a full subcategory $\mathrm{Mod}^{r,\phi}_{/\Sigma}$ of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ to be the category consisting of M such that

- the Σ-module M is free of finite rank and generated by the image of ϕ_r,
- the quotient $M/\mathrm{Fil}^r M$ is p-torsion free.

Then we see that for $\hat{M} \in \mathrm{Mod}^{r,\phi}_{/\Sigma}$, the quotient $\hat{M}/p^n\hat{M}$ is naturally considered as an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$.

The natural ring isomorphism $\Sigma_1/\mathrm{Fil}^p\Sigma_1 \cong \tilde{S}_1$ defines a functor $T_{0,\Sigma} : \mathrm{Mod}^{r,\phi}_{/\Sigma_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ by $M \mapsto M/\mathrm{Fil}^p\Sigma_1 \cdot M$. Then just as in the case of the functor $T_0 : \mathrm{Mod}^{r,\phi}_{/S_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ ([4, Proposition 2.2.2.1]), we can show the following lemma.

Lemma 3.1. *The functor $T_{0,\Sigma} : \mathrm{Mod}^{r,\phi}_{/\Sigma_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ is an equivalence of categories.*

On the other hand, [6, Proposition 2.2.1.3] and Nakayama's lemma show the following.

Lemma 3.2. *Let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_1}$ of rank d over Σ_1. Then there exists a basis $\{e_1, \ldots, e_d\}$ of M such that $\mathrm{Fil}^r M = \Sigma_1 u^{r_1} e_1 \oplus \cdots \oplus \Sigma_1 u^{r_d} e_d + \mathrm{Fil}^p\Sigma_1 \cdot M$ for some integers $r_1, \ldots, r_d$ with $0 \le r_i \le er$ for any i.*

Then we can show the following lemma just as in the proof of [6, Lemme 2.3.1.3].

Lemma 3.3. *The functor*

$$M \mapsto \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, A_{\mathrm{crys},\infty})$$

from $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ to the category of G_{K_∞}-modules is exact.

For $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_1}$, we can show as in the case of the category $\mathrm{Mod}^{r,\phi}_{/S_1}$ that there is an isomorphism of G_{K_1}-modules

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, (\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{PD}}) \to \mathrm{Hom}_{\tilde{S}_1,\mathrm{Fil}^r,\phi_r}(T_{0,\Sigma}(M), \tilde{\mathcal{O}}_{\bar{K}}),$$

where $\tilde{\mathcal{O}}_{\bar{K}}$ is considered as an object of $'\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ by the natural isomorphism

$$(\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{PD}}/\mathrm{Fil}^p(\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{PD}} \to \tilde{\mathcal{O}}_{\bar{K}}.$$

Thus [6, Lemme 2.3.1.2] implies the following.

Lemma 3.4. *For $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_1}$, we have*

$$\#\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, (\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{PD}}) = p^d,$$

where $d = \dim_{\Sigma_1} M$.

For the category $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$, we have the following lemma.

Lemma 3.5. *Let M be in $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$. Then there exists $\alpha_1, \ldots, \alpha_d \in \mathrm{Fil}^r M$ such that $\mathrm{Fil}^r M = \Sigma\alpha_1 + \cdots + \Sigma\alpha_d + \mathrm{Fil}^p\Sigma \cdot M$ and the elements $e_1 = \phi_r(\alpha_1), \ldots, e_d = \phi_r(\alpha_d)$ form a system of generators of M.*

Proof. By induction and Lemma 3.2, we may assume that there exists an exact sequence of the category $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$

$$0 \to M' \to M \to M'' \to 0$$

such that the lemma holds for M' and M''. Let $\alpha'_1, \ldots, \alpha'_{l'}$ (*resp.* $\alpha''_1, \ldots, \alpha''_{l''}$) be elements of $\mathrm{Fil}^r M'$ (*resp.* $\mathrm{Fil}^r M''$) as in the lemma. Let $\alpha_l \in \mathrm{Fil}^r M$ be a lift of α''_l. Then the elements $\alpha'_1, \ldots, \alpha'_{l'}, \alpha_1, \ldots, \alpha_{l''}$ satisfy the condition in the lemma for M. □

Corollary 3.6. *Let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ and $C \in M_d(\Sigma)$ be a matrix satisfying*

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)C$$

with the notation of the previous lemma. Let A be an object of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Then a Σ-linear homomorphism $f : M \to A$ preserving Fil^r also commutes with ϕ_r if and only if

$$\phi_r(f(e_1, \ldots, e_d)C) = (f(e_1), \ldots, f(e_d)).$$

Proof. Suppose that the latter condition holds. Then we have $\phi_r(f(\alpha_i)) = f(\phi_r(\alpha_i))$ for any i. We only have to check the equality $\phi_r \circ f = f \circ \phi_r$ on $\mathrm{Fil}^p\Sigma \cdot M$. Suppose that this equality holds on the submodule $p^{l+1}\mathrm{Fil}^p\Sigma \cdot M$. For $m \in M$, we can take $m' \in \mathrm{Fil}^p\Sigma \cdot M$ such that $E(u)^r m = \sum_i s_i\alpha_i + m'$. Let s be in $\mathrm{Fil}^p\Sigma$. Then we have

$$f(\phi_r(p^l sm)) = p^l\phi_r(s)c^{-r}\sum_i \phi(s_i)f(\phi_r(\alpha_i)) + p^l\phi_r(s)c^{-r}f(\phi_r(m')).$$

Since $\phi_r(\mathrm{Fil}^p\Sigma) \subseteq p\Sigma$, this equals to $\phi_r(f(p^l sm))$ by assumption. Thus the lemma follows by induction. □

Corollary 3.7. *Let M and A be as above and $J \subseteq \mathrm{Fil}^r A$ be a Σ-submodule of A such that $\phi_r(J) \subseteq J$. We can consider the Σ-module A/J naturally as an object of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Suppose that for any $x \in J$, there exists $t \in \mathbb{Z}_{\geq 0}$ such that $\phi_r^t(x) = 0$. Then the natural homomorphism of abelian groups*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, A) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, A/J)$$

is an isomorphism.

Proof. The proof is similar to [3, Subsection 2.2]. We consider the Σ-submodule J as an object of the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ by putting $\mathrm{Fil}^r J = J$. By devissage, it is enough to show that, for any $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_1}$, we have $\mathrm{Ext}_{{}'\mathrm{Mod}^{r,\phi}_{/\Sigma}}(M, J) = 0$ and the map in the corollary is an isomorphism. For the first assertion, let

$$0 \longrightarrow J \longrightarrow \mathcal{E} \longrightarrow M \longrightarrow 0$$

be an extension in the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Let e_i, α_i and C be as in Corollary 3.6 such that $e_1,\ldots,e_d$ form a basis of M. Let $\hat{e}_i\in\mathcal{E}$ be a lift of $e_i\in M$. Then we have $(\hat{e}_1,\ldots,\hat{e}_d)C\in(\mathrm{Fil}^r\mathcal{E})^{\oplus d}$ and

$$\phi_r((\hat{e}_1,\ldots,\hat{e}_d)C)=(\hat{e}_1+\delta_1,\ldots,\hat{e}_d+\delta_d)$$

for some $\delta_1,\ldots,\delta_d\in J$. On the other hand, there exists a unique d-tuple $(x_1,\ldots,x_d)\in J^{\oplus d}$ satisfying the equation

$$\phi_r((\hat{e}_1+x_1,\ldots,\hat{e}_d+x_d)C)=(\hat{e}_1+x_1,\ldots,\hat{e}_d+x_d).$$

Indeed, the d-tuple

$$\sum_{i=0}^{t}(\phi_r^i(\delta_1),\ldots,\phi_r^i(\delta_d))\phi(C)\cdots\phi^{i-1}(C)\phi^i(C)$$

is stable for sufficiently large t by assumption and this limit gives a unique solution of the equation. Then we have

$$(p(\hat{e}_1+x_1),\ldots,p(\hat{e}_d+x_d))=\phi_r(p(\hat{e}_1+x_1),\ldots,p(\hat{e}_d+x_d))\phi(C).$$

Since the d-tuple on the left-hand side is contained in $J^{\oplus d}$, we see that this d-tuple is zero and $e_i\mapsto\hat{e}_i+x_i$ defines a section $M\to\mathcal{E}$. We can prove the second assertion similarly. $\square$

Next we show that the two categories $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ and $\mathrm{Mod}^{r,\phi}_{/S_\infty}$ are in fact equivalent. For $M\in\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$, we associate to it an S-module $\mathcal{M}$ by setting $\mathcal{M}=S\otimes_\Sigma M$. We also define its S-submodule $\mathrm{Fil}^r\mathcal{M}$ by

$$\mathrm{Fil}^r\mathcal{M}=\mathrm{Ker}(\mathcal{M}=S\otimes_\Sigma M\to S/\mathrm{Fil}^r S\otimes_\Sigma M/\mathrm{Fil}^r M\simeq M/\mathrm{Fil}^r M),$$

where the last isomorphism is induced by the natural isomorphisms of W-algebras

$$W[u]/(E(u)^r)\to\Sigma/\mathrm{Fil}^r\Sigma\to S/\mathrm{Fil}^r S.$$

These associations induce two functors from $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ to the category of S-modules, $M\mapsto\mathcal{M}$ and $M\mapsto\mathrm{Fil}^r\mathcal{M}$. Since the rings S and $W[u]/(E(u)^r)$ are p-torsion free, we have $\mathrm{Tor}_1^\Sigma(\Sigma_1,S)=\mathrm{Tor}_1^\Sigma(\Sigma_1,\Sigma/\mathrm{Fil}^r\Sigma)=0$ and thus $\mathrm{Tor}_1^\Sigma(M,S)=\mathrm{Tor}_1^\Sigma(M,\Sigma/\mathrm{Fil}^r\Sigma)=0$ for any $M\in\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$. Hence we see that these two functors are exact.

We define $\phi_r:\mathrm{Fil}^r\mathcal{M}\to\mathcal{M}$ as follows. Note that $\mathrm{Fil}^r S\otimes_\Sigma M\subseteq\mathcal{M}$ and $\mathrm{Fil}^r\mathcal{M}$ is equal to $\mathrm{Fil}^r S\otimes_\Sigma M+\mathrm{Im}(S\otimes_\Sigma\mathrm{Fil}^r M\to\mathcal{M})$. Set $\phi'_r:\mathrm{Fil}^r S\otimes_\Sigma M\to\mathcal{M}$ to be $\phi'_r=\phi_r\otimes\phi$.

Lemma 3.8. *The map $\phi\otimes\phi_r:S\otimes_\Sigma\mathrm{Fil}^r M\to\mathcal{M}$ induces a ϕ-semilinear map $\phi''_r:\mathrm{Im}(S\otimes_\Sigma\mathrm{Fil}^r M\to\mathcal{M})\to\mathcal{M}$.*

Proof. Let $z=\sum_i s_i\otimes m_i$ be in $S\otimes_\Sigma\mathrm{Fil}^r M$ with $s_i\in S$ and $m_i\in\mathrm{Fil}^r M$. Let $\bar{z}$ be its image in $\mathcal{M}$ and suppose that $\bar{z}=0$. Write $s_i=s'_i+s''_i$ with $s'_i\in\Sigma$ and $s''_i\in\mathrm{Fil}^p S$. Since we have an isomorphism $\mathcal{M}/\mathrm{Fil}^r S\cdot\mathcal{M}\simeq M/\mathrm{Fil}^r\Sigma\cdot M$,

we can find elements $s^{(j)} \in \mathrm{Fil}^r\Sigma$ and $m^{(j)} \in M$ such that the equality $\sum_i s'_i m_i = \sum_j s^{(j)} m^{(j)}$ holds in M. Then we have

$$0 = \bar{z} = \sum_i 1 \otimes s'_i m_i + \sum_i s''_i \otimes m_i = \sum_j s^{(j)} \otimes m^{(j)} + \sum_i s''_i \otimes m_i$$

in $\mathcal{M}$. On the other hand, the element $(\phi \otimes \phi_r)(z) \in \mathcal{M}$ is equal to

$$\sum_j 1 \otimes \phi_r(s^{(j)} m^{(j)}) + \sum_i \phi(s''_i) \otimes \phi_r(m_i).$$

Since $\phi = p^r \phi_r$, this equals $\phi'_r(\sum_j s^{(j)} \otimes m^{(j)} + \sum_i s''_i \otimes m_i) = 0$. $\square$

Lemma 3.9. *The maps ϕ'_r and ϕ''_r patch together and define a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$.*

Proof. Since ϕ'_r and ϕ''_r coincide on $\mathrm{Im}(\mathrm{Fil}^r S \otimes_\Sigma \mathrm{Fil}^r M \to \mathcal{M})$, it is enough to show that $1 \otimes \phi_r(m) = \phi'_r(\sum_i s_i \otimes m_i)$ for any $m \in \mathrm{Fil}^r M$, $s_i \in \mathrm{Fil}^r S$ and $m_i \in M$ satisfying $1 \otimes m = \sum_i s_i \otimes m_i$ in $\mathcal{M}$. As in the proof of Lemma 3.8, the element m can be written as $m = \sum_j s^{(j)} m^{(j)}$ for some $s^{(j)} \in \mathrm{Fil}^r\Sigma$ and $m^{(j)} \in M$. By assumption, we have $\sum_i s_i \otimes m_i = \sum_j s^{(j)} \otimes m^{(j)}$ in $\mathrm{Fil}^r S \otimes_\Sigma M$. Hence the lemma follows. $\square$

Then we see that this construction defines a functor $\mathcal{M}_{\Sigma_\infty} : \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$.

Lemma 3.10. *The functor $\mathcal{M}_{\Sigma_\infty}$ induces an equivalence of categories $\mathrm{Mod}^{r,\phi}_{/\Sigma_1} \to \mathrm{Mod}^{r,\phi}_{/S_1}$.*

Proof. Consider the diagram of functors

$$\begin{tikzcd} \mathrm{Mod}^{r,\phi}_{/\Sigma_1} \arrow[r, "\mathcal{M}_{\Sigma_\infty}"] \arrow[dr, "T_{0,\Sigma}"'] & \mathrm{Mod}^{r,\phi}_{/S_1} \arrow[d, "T_0"] \\ & \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}. \end{tikzcd}$$

From the definition, we see that this diagram is commutative. By Lemma 3.1, the downward arrows are equivalences of categories. Thus the lemma follows. $\square$

Then a devissage argument as in [15, Proposition 1.1.11] shows the following corollary.

Corollary 3.11. *The functor $\mathcal{M}_{\Sigma_\infty} : \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$ is fully faithful.*

To show the essential surjectivity of the functor $\mathcal{M}_{\Sigma_\infty}$, we define another functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ which is defined in a similar way to the functor $\mathcal{M}_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$. For an $\mathfrak{S}$-module $\mathfrak{M}$ in $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, we associate to it a Σ-module $M \in {}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ as follows:

- $M = \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$,
- $\mathrm{Fil}^r M = \mathrm{Ker}(M \overset{1\otimes\phi}{\to} \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \to (\Sigma/\mathrm{Fil}^r \Sigma) \otimes_{\mathfrak{S}} \mathfrak{M})$,
- $\phi_r : \mathrm{Fil}^r M \overset{1\otimes\phi}{\to} \mathrm{Fil}^r \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_r\otimes 1}{\to} \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M} = M$.

We can check that this defines an exact functor $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ as in the proof of [15, Proposition 1.1.11]. We let this functor be denoted by $M_{\mathfrak{S}_\infty}$.

Lemma 3.12. *The diagram of functors*

$$\begin{tikzcd} \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty} \arrow[r, "M_{\mathfrak{S}_\infty}"] \arrow[dr, "\mathcal{M}_{\mathfrak{S}_\infty}"'] & \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty} \arrow[d, "\mathcal{M}_{\Sigma_\infty}"] \\ & \mathrm{Mod}^{r,\phi}_{/S_\infty} \end{tikzcd}$$

is commutative.

Proof. For $\mathfrak{M} \in \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_\infty}$, put $M = M_{\mathfrak{S}_\infty}(\mathfrak{M})$ and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}_\infty}(\mathfrak{M})$. Then $\mathcal{M} = S \otimes_\Sigma M$ as an S-module. Let $\mathrm{Fil}^r \mathcal{M}$ and $\phi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ denote the filtration and Frobenius structure defined by the functor $\mathcal{M}_{\mathfrak{S}_\infty}$. We also let $\hat{\mathrm{Fil}}^r \mathcal{M}$ and $\hat{\phi}_r : \hat{\mathrm{Fil}}^r \mathcal{M} \to \mathcal{M}$ denote those defined by $\mathcal{M}_{\Sigma_\infty}$.

The S-module $\mathrm{Fil}^r \mathcal{M}$ contains $\hat{\mathrm{Fil}}^r \mathcal{M}$. Conversely, let z be an element of $\mathrm{Fil}^r \mathcal{M}$. Note that $\mathrm{Fil}^p S \cdot \mathcal{M} \subseteq \hat{\mathrm{Fil}}^r \mathcal{M}$. Thus, to show $z \in \hat{\mathrm{Fil}}^r \mathcal{M}$, we may assume that $z \in \mathrm{Im}(M \to \mathcal{M})$. Then the commutative diagram whose right vertical arrow is an isomorphism

$$\begin{array}{ccccc} M = \Sigma \otimes_{\phi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{1\otimes\phi} & \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & \Sigma/\mathrm{Fil}^r \Sigma \otimes_{\mathfrak{S}} \mathfrak{M} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M} = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{1\otimes\phi} & S \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & S/\mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \end{array}$$

implies that $z \in \mathrm{Im}(\mathrm{Fil}^r M \to \mathrm{Fil}^r \mathcal{M}) \subseteq \hat{\mathrm{Fil}}^r \mathcal{M}$ and hence $\mathrm{Fil}^r \mathcal{M} = \hat{\mathrm{Fil}}^r \mathcal{M}$. From the definition, we also can show $\phi_r = \hat{\phi}_r$. This implies the lemma. □

Proposition 3.13. *The functor $\mathcal{M}_{\Sigma_\infty} : \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty} \to \mathrm{Mod}^{r,\phi}_{/S_\infty}$ is an equivalence of categories.*

Proof. Since the functor $\mathcal{M}_{\mathfrak{S}_\infty}$ is an equivalence of categories for $p \geq 3$ ([9, Theorem 2.3.1]), Corollary 3.11 and Lemma 3.12 imply the proposition in this case. For $r = 0$, put $\kappa_\Sigma = \kappa \cap \Sigma$, where $\kappa = \mathrm{Ker}(S \to W)$. Then, by using a natural isomorphism $\Sigma \simeq W[[u, u^{ep}/p]]$, we can show that the functor $M \mapsto M/\kappa_\Sigma M$ defines an equivalence of categories $\mathrm{Mod}^{0,\phi}_{/\Sigma_\infty} \to \mathrm{Mod}^{\phi}_{/W_\infty}$, as

in the case of the category $\mathrm{Mod}^{0,\phi}_{/S_\infty}$. Since the diagram

$$\begin{tikzcd} \mathrm{Mod}^{0,\phi}_{/\Sigma_\infty} \arrow[r, "\mathcal{M}_{\Sigma_\infty}"] \arrow[dr] & \mathrm{Mod}^{0,\phi}_{/S_\infty} \arrow[d] \\ & \mathrm{Mod}^{\phi}_{/W_\infty} \end{tikzcd}$$

is commutative and the downward arrows are equivalences of categories, the proposition follows also for $p = 2$. □

Remark 3.14. We can also define a fully faithful functor $\mathcal{M}_\Sigma : \mathrm{Mod}^{r,\phi}_{/\Sigma} \to \mathrm{Mod}^{r,\phi}_{/S}$ in a similar way to $\mathcal{M}_{\Sigma_\infty}$ and prove that this is an equivalence of categories. Indeed, the claim for $p \geq 3$ follows from [9, Theorem 2.2.1]. Let $\mathcal{M}$ be in $\mathrm{Mod}^{0,\phi}_{/S}$ and $e_1, \dots, e_d$ be a basis of $\mathcal{M}$ over S. Let $C \in GL_d(S)$ be the matrix such that

$$\phi(e_1, \dots, e_d) = (e_1, \dots, e_d)C.$$

Then the elements $\phi(e_1), \dots, \phi(e_d)$ also form a basis of $\mathcal{M}$ and

$$\phi(\phi(e_1), \dots, \phi(e_d)) = (\phi(e_1), \dots, \phi(e_d))\phi(C).$$

Since $\phi(S) \subseteq \Sigma$, the Σ-module M defined by $M = \Sigma\phi(e_1) \oplus \cdots \oplus \Sigma\phi(e_d)$ is stable under ϕ. Hence we see that $M \in \mathrm{Mod}^{0,\phi}_{/\Sigma}$ and $\mathcal{M} = \mathcal{M}_\Sigma(M)$.

Proposition 3.15. *Let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ and set $\mathcal{M} = \mathcal{M}_{\Sigma_\infty}(M)$. Then there exists a natural isomorphism of G_{K_∞}-modules*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, A_{\mathrm{crys},\infty}) \to \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, A_{\mathrm{crys},\infty}).$$

Moreover, this induces for any n an isomorphism of G_{K_n}-modules

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})) \to \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})).$$

Proof. By definition, $\mathcal{M} = S \otimes_\Sigma M$ and we have a natural isomorphism

$$\mathrm{Hom}_\Sigma(M, A_{\mathrm{crys},\infty}) \to \mathrm{Hom}_S(\mathcal{M}, A_{\mathrm{crys},\infty}).$$

From the definition, we can check that this isomorphism induces the map in the proposition, which is injective. To prove the bijectivity, by devissage we may assume that $pM = 0$. Then both sides of this injection have the same cardinality by Lemma 3.4 and the first assertion follows. Since the sequence

$$0 \longrightarrow W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}) \longrightarrow A_{\mathrm{crys},\infty} \xrightarrow{p^n} A_{\mathrm{crys},\infty} \longrightarrow 0$$

of the category $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ is exact, the first assertion implies the second one. □

4. A method of Abrashkin

In this section, we study the G_{K_n}-module $\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}))$ following Abrashkin ([3]).

Let p and $0 \leq r < p-1$ be as before. We fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$ in $\bar{K}$ such that $\zeta_p \neq 1$ and $\zeta_{p^n} = \zeta_{p^{n+1}}^p$ for any n, and set an element $\underline{\varepsilon}$ of R to be $(\zeta_{p^n})_{n\in\mathbb{Z}_{\geq 0}}$. Then the elements $[\underline{\varepsilon}]-1$ and $[\underline{\varepsilon}^{1/p}]-1$ are topologically nilpotent in $W(R)$. The element of $W(R)$

$$t = ([\underline{\varepsilon}]-1)/([\underline{\varepsilon}^{1/p}]-1) = 1 + [\underline{\varepsilon}^{1/p}] + [\underline{\varepsilon}^{1/p}]^2 + \cdots + [\underline{\varepsilon}^{1/p}]^{p-1}$$

is a generator of the principal ideal $\mathrm{Ker}(\theta)$. We define an element $a \in W(R)$ to be

$$a = \begin{cases} \sum_{k=1}^{p-2} p^{-1}((-1)^{p-1-k}{}_{p-1}C_k - 1)[\underline{\varepsilon}^{1/p}]^k & (p \geq 3) \\ -1 & (p=2), \end{cases}$$

where ${}_{p-1}C_k = (p-1)!/(k!(p-1-k)!)$ is the binomial coefficient. Note that the coefficient of $[\underline{\varepsilon}^{1/p}]^k$ in each term is an integer. The element a is invertible in the ring $W(R)$, since $\theta(a) = (\zeta_p - 1)^{p-1}/p \in \mathcal{O}_{\mathbb{C}}^\times$ and the ideal $\mathrm{Ker}(\theta)$ is topologically nilpotent in $W(R)$.

The element $Z = ([\underline{\varepsilon}]-1)^{p-1}/p$ of A_{crys} is topologically nilpotent and we have $\phi(t) = p(Z - \phi(a))$. Consider the formal power series ring $W(R)[[u']]$ with the (t,u')-adic topology and the continuous ring homomorphism $W(R)[[u']] \to A_{\mathrm{crys}}$ which sends u' to Z. Let $\hat{A}$ denote the image of this homomorphism. Then we see that the ring $\hat{A}$ is (t,Z)-adically complete. Since we have $Z = at^{p-1} + t^p/p$, the element t^p/p of A_{crys} is contained in the subring $\hat{A}$ and topologically nilpotent in this subring. Hence we can consider the ring $\hat{A}$ as a Σ-algebra by $u \mapsto [\underline{\pi}]$. Put $\mathrm{Fil}^i\hat{A} = (t^i, Z)$ for $0 \leq i \leq p-1$. The Frobenius endomorphism ϕ of A_{crys} preserves $\hat{A}$ and satisfies $\phi(\mathrm{Fil}^i\hat{A}) \subseteq p^i\hat{A}$ for $0 \leq i \leq p-1$. Set $\phi_r = p^{-r}\phi|_{\mathrm{Fil}^r\hat{A}}$. Then we can consider the ring $\hat{A}$ also as an object of the category ${}'\mathrm{Mod}_{/\Sigma}^{r,\phi}$. Put $\hat{A}_n = \hat{A}/p^n\hat{A}$ and $\hat{A}_\infty = \hat{A}\otimes_W K_0/W$. We include here a proof of the following lemma stated in [3, Subsection 3.2].

Lemma 4.1. *The natural inclusion $W(R) \to \hat{A}$ induces isomorphisms of $W(R)$-algebras $W(R)/(([\underline{\varepsilon}]-1)^{p-1}) \to \hat{A}/(Z)$ and $W_n(R)/(([\underline{\varepsilon}]-1)^{p-1}) \to \hat{A}_n/(Z)$.*

Proof. For a subring B of A_{crys}, put

$$I^{[s]}B = \{x \in B \mid \phi^i(x) \in \mathrm{Fil}^s A_{\mathrm{crys}} \text{ for any } i\}$$

as in [12, Subsection 5.3]. Then we have $I^{[s]}W(R) = ([\underline{\varepsilon}]-1)^s W(R)$ and the natural ring homomorphism

$$W(R)/I^{[s]}W(R) \to A_{\mathrm{crys}}/I^{[s]}A_{\mathrm{crys}}$$

is an injection ([12, Proposition 5.1.3, Proposition 5.3.5]). Since the element Z is contained in the ideal $I^{[p-1]}A_{\mathrm{crys}}$, this injection factors as

$$W(R)/I^{[p-1]}W(R) \to \hat{A}/(Z) \to A_{\mathrm{crys}}/I^{[p-1]}A_{\mathrm{crys}}.$$

Hence the former arrow is an isomorphism and the lemma follows. □

Therefore $\hat{A}/\mathrm{Fil}^r\hat{A}$ is p-torsion free and $p^n\mathrm{Fil}^r\hat{A} = \mathrm{Fil}^r\hat{A} \cap p^n\hat{A}$. Thus we can also consider $\hat{A}_n$ and $\hat{A}_\infty$ as objects of the category $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. The absolute Galois group G_{K_∞} acts naturally on these Σ-modules.

Lemma 4.2. *We have a natural decomposition as an R-module*

$$\hat{A}_1 = R/(t^p) \oplus (Z).$$

Proof. Consider the natural inclusion $W(R) \to \hat{A}$. We claim that this induces an injection $R/(t^p) \to \hat{A}_1$. Let x be in the ring R. If the element $[x] \in W(R)$ is contained in $p\hat{A}$, then its image in $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$ is zero. We have an isomorphism of R-algebras

$$R[Y_1, Y_2, \ldots]/(t^p, Y_1^p, Y_2^p, \ldots) \to A_{\mathrm{crys}}/pA_{\mathrm{crys}}$$

which sends Y_i to the image of $t^{p^i}/p^i!$. Thus the element x is contained in the ideal (t^p). Conversely, if $v_R(x) \geq p$, then we have

$$[x] = w([\underline{\varepsilon}] - 1)^{p-1} + pw'$$

for some $w, w' \in W(R)$ and this implies $[x] \in p\hat{A}$. Now we have the commutative diagram of R-algebras

$$\begin{tikzcd} R/(t^p) \arrow[r] \arrow[dr, "f"'] & \hat{A}_1 \arrow[d] \\ & \hat{A}_1/(Z) \end{tikzcd}$$

and the map $f : R/(t^p) \to \hat{A}_1/(Z)$ is an isomorphism by Lemma 4.1. Hence the lemma follows. □

Since $r < p - 1$, from this lemma we can show the following lemma as in the proof of [6, Lemme 2.3.1.3].

Lemma 4.3. *The functor*

$$M \mapsto \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \hat{A}_\infty)$$

from $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ to the category of G_{K_∞}-modules is exact.

Corollary 4.4. *For any $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$, the natural map*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \hat{A}_\infty) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, A_{\mathrm{crys},\infty})$$

is an isomorphism of G_{K_∞}-modules. Moreover, for any n, we have an isomorphism of G_{K_∞}-modules

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,\hat{A}_n)\to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,A_{\mathrm{crys}}/p^nA_{\mathrm{crys}}).$$

Proof. Let us prove the first assertion. By Lemma 3.3 and Lemma 4.3, we may assume $pM=0$. Consider the commutative diagram of rings

$$\begin{tikzcd} \hat{A}_1 \arrow[r] \arrow[dr] & A_{\mathrm{crys}}/pA_{\mathrm{crys}} \arrow[d] \\ & R/(t^{p-1}) \end{tikzcd}$$

whose downward arrows are defined by modulo Fil^{p-1} of the rings $\hat{A}_1$ and $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$, respectively. Since $r<p-1$, we have $\phi_r(\mathrm{Fil}^{p-1}\hat{A}_1)=0$ and similarly for the ring $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. Thus these two surjections induce on the ring $R/(t^{p-1})$ the same structure of a filtered ϕ_r-module over Σ. By Corollary 3.7, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,\hat{A}_1) \arrow[r] \arrow[dr] & \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,A_{\mathrm{crys}}/pA_{\mathrm{crys}}) \arrow[d] \\ & \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,R/(t^{p-1})) \end{tikzcd}$$

whose downward arrows are isomorphisms. This concludes the proof of the first assertion. Since we have an exact sequence

$$0\longrightarrow \hat{A}_n\longrightarrow \hat{A}_\infty \xrightarrow{p^n} \hat{A}_\infty\longrightarrow 0$$

in the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$, the second assertion follows. □

Since the ideal (Z) of $\hat{A}_n$ satisfies the condition of Corollary 3.7, the Σ-algebra $\hat{A}_n/(Z)$ is naturally considered as an object of ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. We also give the ring $W_n(R)/(([\underline{\varepsilon}]-1)^{p-1})$ the structures of a Σ-algebra and a filtered ϕ_r-module over Σ induced from those of $\hat{A}_n/(Z)$ by the isomorphism in Lemma 4.1. The map

$$\Sigma\to W_n(R)/(([\underline{\varepsilon}]-1)^{p-1})$$

sends the element $u\in\Sigma$ to the image of $[\underline{\pi}]$ in the ring on the right-hand side. Put $v=t/E([\underline{\pi}])\in W(R)^\times$. As for the element $Y\in\Sigma$, the equality

$$Y=-av^{-1}E([\underline{\pi}])^{p-1}+v^{-p}Z$$

holds in $\hat{A}$. Hence the above homomorphism sends the element Y to the image of $-av^{-1}E([\underline{\pi}])^{p-1}$.

Consider the surjective ring homomorphism

$$R\to\tilde{\mathcal{O}}_{\bar{K}}$$
$$x=(x_0,x_1,\dots)\mapsto x_n$$

and the induced surjection $\beta_n : W_n(R) \to W_n(\tilde{\mathcal{O}}_{\bar{K}})$. Let

$$J = \{(x_0, \ldots, x_{n-1}) \in W_n(R) \mid v_R(x_i) \geq p^n \text{ for any } i\}$$

be the kernel of the latter surjection.

Lemma 4.5. *The ideal J is contained in the ideal $(([\underline{\varepsilon}] - 1)^{p-1})$ of the ring $W_n(R)$.*

Proof. Write the element $([\underline{\varepsilon}] - 1)^{p-1}$ also as $x = (x_0, \ldots, x_{n-1}) \in W_n(R)$ with $v_R(x_0) = p$. Take an element $z = (z_0, \ldots, z_{n-1})$ of the ideal J. We construct $y \in W_n(R)$ such that $xy = z$. By induction, it is enough to show that if $z_0 = \cdots = z_{i-1} = 0$ for some $0 \leq i \leq n-1$ and $(x_0, \ldots, x_i)(0, \ldots, 0, y_i) = (0, \ldots, 0, z_i)$ in $W_{i+1}(R)$, then $x(0, \ldots, 0, y_i, 0, \ldots, 0) \in J$. Let us write this element as $(0, \ldots, 0, w_i, \ldots, w_{n-1})$ with $w_i = z_i$. We have $v_R(y_i) \geq p^n - p^{i+1}$. In the ring of Witt vectors $W_n(\mathbb{F}_p[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}])$, the k-th entry of the vector

$$(X_0, \ldots, X_{n-1})(0, \ldots, 0, Y_i, 0, \ldots, 0)$$

is $X_{k-i}^{p^i} Y_i^{p^{k-i}}$ for any $k \geq i$. Thus we have $v_R(w_k) \geq p^n$. □

Note that the elements $[\zeta_{p^n}] - 1$ and $[\zeta_{p^{n+1}}] - 1$ are nilpotent in $W_n(\tilde{\mathcal{O}}_{\bar{K}})$. By the above lemma, we have an isomorphism of rings

$$W_n(R)/(([\underline{\varepsilon}] - 1)^{p-1}) \to W_n(\tilde{\mathcal{O}}_{\bar{K}})/(([\zeta_{p^n}] - 1)^{p-1}).$$

We let $\bar{A}_{n,p-1}$ denote the ring on the right-hand side and give the ring $\bar{A}_{n,p-1}$ the structure of a filtered ϕ_r-module over Σ induced by this isomorphism.

For an algebraic extension F of K, we put

$$\mathfrak{b}_F = \{x \in \mathcal{O}_F \mid v_K(x) > er/(p-1)\}.$$

Note that the ring $\mathcal{O}_F/\mathfrak{b}_F$ is killed by p. We consider the ring of Witt vectors $W_n(\mathcal{O}_F/\mathfrak{b}_F)$ as a $W_n(\mathcal{O}_F)$-algebra by the natural ring surjection $W_n(\mathcal{O}_F) \to W_n(\mathcal{O}_F/\mathfrak{b}_F)$ and as a W_n-algebra by twisting the natural action by σ^{-n}, as before. For a ring B and its ideal I, we define an ideal $W_n(I)$ of the ring $W_n(B)$ to be

$$W_n(I) = \{(x_0, \ldots, x_{n-1}) \in W_n(B) \mid x_i \in I \text{ for any } i\}.$$

Put $F_n = K_n(\zeta_{p^{n+1}})$. For an algebraic extension F of F_n in $\bar{K}$, the elements $[\zeta_{p^n}] - 1$ and $[\zeta_{p^{n+1}}] - 1$ of $W_n(m_F)$ are topologically nilpotent non-zero divisors in $W_n(\mathcal{O}_F)$. Let the ring

$$W_n(\mathcal{O}_F/\mathfrak{b}_F)/([\zeta_{p^n}] - 1)^r W_n(m_F/\mathfrak{b}_F)$$

be denoted by $\bar{A}_{n,F,r+}$. We also put $\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+}$.

Lemma 4.6. *The ideal $([\zeta_{p^n}] - 1)^r W_n(m_F)$ of $W_n(\mathcal{O}_F)$ contains the ideal $W_n(\mathfrak{b}_F)$ for any $r \in \{0, \ldots, p-2\}$. We also have $(([\zeta_{p^n}]-1)^{p-1}) \supseteq W_n(p\mathcal{O}_F)$.*

Proof. The proof is similar to the proof of Lemma 4.5. Let us show the first assertion. Since this is trivial for $r = 0$, we may assume $r \geq 1$. Put $x = (x_0, \ldots, x_{n-1}) = ([\zeta_{p^n}] - 1)^r \in W_n(\mathcal{O}_F)$. Then we have $v_p(x_0) = r/(p^{n-1}(p-1))$. By induction, it is enough to show that for $0 \leq i \leq n-1$, if $(x_0, \ldots, x_i)(0, \ldots, 0, y_i) \in W_{i+1}(\mathfrak{b}_F)$, then $y_i \in m_F$ and $x(0, \ldots, 0, y_i, 0, \ldots, 0) \in W_n(\mathfrak{b}_F)$. By assumption, we have

$$v_p(y_i) > \frac{r}{p-1}(1 - \frac{1}{p^{n-i-1}}) \geq 0.$$

Put $(0, \ldots, 0, w_i, \ldots, w_{n-1}) = x(0, \ldots, 0, y_i, 0, \ldots, 0)$. We show $w_l \in \mathfrak{b}_F$ for any l by induction. Indeed, let us suppose that $w_l \in \mathfrak{b}_F$ for any $i \leq l \leq k-1$ with some $i+1 \leq k \leq n-1$. We have the equality

$$p^i y_i^{p^{k-i}}(x_0^{p^k} + p x_1^{p^{k-1}} + \cdots + p^k x_k) = (p^i w_i^{p^{k-i}} + p^{i+1} w_{i+1}^{p^{k-i-1}} + \cdots + p^k w_k).$$

Since $r \geq 1$, we have $(p^{k-l} - 1)r/(p-1) \geq k - l$ for $0 \leq l \leq k-1$. This implies $v_p(p^l w_l^{p^{k-l}}) > k + r/(p-1)$ for $0 \leq l \leq k-1$. The valuation of the left-hand side of the above equality also satisfies this inequality. Thus we have $v_p(w_k) > r/(p-1)$ and the assertion follows. We can show the second assertion similarly. $\square$

By this lemma, the natural surjections of rings

$$\begin{aligned} &W_n(\mathcal{O}_F)/([\zeta_{p^n}] - 1)^r W_n(m_F) \\ &\qquad \to W_n(\mathcal{O}_F/p\mathcal{O}_F)/([\zeta_{p^n}] - 1)^r W_n(m_F/p\mathcal{O}_F) \to \bar{A}_{n,F,r+} \end{aligned}$$

are isomorphisms. Then we see that the natural injection $F \to \bar{K}$ induces an injection of rings $\bar{A}_{n,F,r+} \to \bar{A}_{n,r+}$.

Write Z_n for the image of the element Z of A_{crys} in $W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})$. Then we have a commutative diagram of Σ-algebras

$$\begin{tikzcd}
 & \hat{A}_n \arrow[r] \arrow[dd] & A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \arrow[d, "\wr"] \\
 & & W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}}) \arrow[dd] \\
W_n(R)/(([\underline{\varepsilon}] - 1)^{p-1}) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \hat{A}_n/(Z) \arrow[dr] & \\
\bar{A}_{n,p-1} \arrow[rr] \arrow[d] & & W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})/(Z_n), \\
\bar{A}_{n,r+} & &
\end{tikzcd}$$

where all the vertical arrows are surjections satisfying the condition of Corollary 3.7. Hence this is also a commutative diagram in $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. Note

that these rings and homomorphisms are independent of the choice of a system $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$. We also note that $\mathrm{Fil}^r\bar{A}_{n,r+} = E([\pi_n])^r\bar{A}_{n,r+}$ and $\phi_r(E([\pi_n])^r y) = c^r\phi(y)$ for any $y \in \bar{A}_{n,r+}$, where ϕ denotes the Frobenius endomorphism of $\bar{A}_{n,r+}$ induced from that of the ring $W_n(\mathcal{O}_{\bar{K}}/\mathfrak{b}_{\bar{K}})$. Moreover, let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$. Then, by Corollary 3.7 and Corollary 4.4, we have a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \bar{A}_{n,r+}).$$

Next we investigate the module on the right-hand side of this isomorphism, and prove this is in fact an isomorphism of G_{F_n}-modules. Consider the element $E([\pi_n]) \in W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$ and let us fix its lift $\hat{\gamma} \in W_n(\mathcal{O}_{F_n})$ by the natural surjection $W_n(\mathcal{O}_{F_n}) \to W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Let $a \in W(R)^\times$ and $v = t/E([\underline{\pi}]) \in W(R)^\times$ as before. We let a_n, t_n and v_n denote the images of a, t and v by the surjection $W(R) \to W_n(\tilde{\mathcal{O}}_{\bar{K}})$ induced by β_n, respectively. The elements a_n and t_n of the ring $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ are contained in the subring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. We abusively let them also denote their images by the natural surjections $W_n(\tilde{\mathcal{O}}_{\bar{K}}) \to W_n(\mathcal{O}_{\bar{K}}/\mathfrak{b}_{\bar{K}}) \to \bar{A}_{n,r+}$.

Lemma 4.7. *The element*

$$\hat{t}_n = 1 + [\zeta_{p^{n+1}}] + [\zeta_{p^{n+1}}]^2 + \cdots + [\zeta_{p^{n+1}}]^{p-1} = \frac{[\zeta_{p^n}] - 1}{[\zeta_{p^{n+1}}] - 1}$$

is divisible by $\hat{\gamma}$ in the ring $W_n(\mathcal{O}_{F_n})$. In particular, $\hat{\gamma}$ is a non-zero divisor of the ring $W_n(\mathcal{O}_{\bar{K}})$.

Proof. It is enough to show the divisibility in the ring $W_n(\mathcal{O}_{\bar{K}})$. Note that the element t_n is also the image of $\hat{t}_n$ by the natural map $W_n(\mathcal{O}_{F_n}) \to W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$. Let $\hat{v}_n$ be a lift of v_n by the natural surjection $W_n(\mathcal{O}_{\bar{K}}) \to W_n(\tilde{\mathcal{O}}_{\bar{K}})$. Then we have $\hat{t}_n - \hat{\gamma}\hat{v}_n \in W_n(p\mathcal{O}_{\bar{K}})$. By Lemma 4.6, there exists $\hat{y} \in W_n(m_{\bar{K}})$ such that $\hat{t}_n - \hat{\gamma}\hat{v}_n = \hat{t}_n\hat{y}$. Hence we have $\hat{t}_n(1-\hat{y}) = \hat{\gamma}\hat{v}_n$. Since $\hat{y}$ is topologically nilpotent in the ring $W_n(\mathcal{O}_{\bar{K}})$, the element $1-\hat{y}$ is invertible and the lemma follows. □

Lemma 4.8. *The image of $Y \in \Sigma$ in the ring $\bar{A}_{n,r+}$ (resp. $\bar{A}_{n,p-1}$) is contained in its subring $\bar{A}_{n,F_n,r+}$ (resp. $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})/(([\zeta_{p^n}]-1)^{p-1})$).*

Proof. We have the equality

$$E([\pi_n])v_n = t_n = 1 + [\zeta_{p^{n+1}}] + [\zeta_{p^{n+1}}]^2 + \cdots + [\zeta_{p^{n+1}}]^{p-1}$$

in the ring $W_n(\tilde{\mathcal{O}}_{\bar{K}})$. Note that any element $v'_n \in W_n(\tilde{\mathcal{O}}_{\bar{K}})$ satisfying the same equality is invertible and thus the elements $(v'_n)^{-1}E([\pi_n])$ are equal to each other. Since $Y = -a_n v_n^{-1}E([\pi_n])^{p-1}$ in the rings $\bar{A}_{n,r+}$ and $\bar{A}_{n,p-1}$, it suffices to construct an element v'_n of the ring $W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$ such that the equality $E([\pi_n])v'_n = t_n$ holds. This follows from Lemma 4.7. □

From this lemma, we see that the natural G_{F_n}-actions on the rings $\bar{A}_{n,p-1}$ and $\bar{A}_{n,r+}$ are compatible with the filtered ϕ_r-module structures over Σ. In

the big commutative diagram above, the lowest horizontal arrow and lower right vertical arrow are G_K-linear by definition. Hence we have shown the following proposition.

Proposition 4.9. *Let M be an object of $\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$. Then the map*

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, W_n^{\mathrm{PD}}(\tilde{\mathcal{O}}_{\bar{K}})) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \bar{A}_{n,r+})$$

is an isomorphism of G_{F_n}-modules.

Let M be as in the proposition. Let $e_1, \ldots, e_d$ be a system of generators of M as in Lemma 3.5 and $C = (c_{i,j}) \in M_d(\Sigma)$ be a matrix representing ϕ_r as in Corollary 3.6. Consider the surjection $\Sigma^{\oplus d} \to M$ defined by $(s_1, \ldots, s_d) \mapsto s_1 e_1 + \cdots + s_d e_d$ and let $(s_{1,1}, \ldots, s_{1,d}), \ldots, (s_{q,1}, \ldots, s_{q,d})$ be a system of generators of its kernel. Then the underlying G_{F_n}-set of the G_{F_n}-module

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \bar{A}_{n,r+})$$

is identified with the set of d-tuples $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ such that the following three conditions hold:

- $s_{l,1}\bar{x}_1 + \cdots + s_{l,d}\bar{x}_d = 0$ for any l,
- $c_{1,i}\bar{x}_1 + \cdots + c_{d,i}\bar{x}_d \in \mathrm{Fil}^r \bar{A}_{n,r+}$ for any i,
- the following equality holds:

$$\begin{cases} \phi_r(c_{1,1}\bar{x}_1 + \cdots + c_{d,1}\bar{x}_d) = \bar{x}_1 \\ \qquad\qquad\vdots \\ \phi_r(c_{1,d}\bar{x}_1 + \cdots + c_{d,d}\bar{x}_d) = \bar{x}_d. \end{cases}$$

We choose lifts $\hat{c}$, $\hat{c}_{i,j}$ and $\hat{s}_{i,j}$ in $W_n(\mathcal{O}_{F_n})$ of the images of c, $c_{i,j}$ and $s_{i,j}$ in $\bar{A}_{n,r+}$ by the natural ring homomorphism

$$W_n(\mathcal{O}_{\bar{K}}) \to W_n(\tilde{\mathcal{O}}_{\bar{K}}) \to W_n(\mathcal{O}_{\bar{K}}/\mathfrak{b}_{\bar{K}}) \to \bar{A}_{n,r+},$$

respectively. Recall that we have already chosen a lift $\hat{\gamma} \in W_n(\mathcal{O}_{F_n})$ of $E([\pi_n]) \in W_n(\mathcal{O}_{F_n}/p\mathcal{O}_{F_n})$.

Fix a polynomial $\Phi_i \in \mathbb{Z}[X_0, \ldots, X_{n-1}]$ such that $\Phi_i \equiv X_i^p \bmod p$. This induces for any commutative ring B a map $\Phi = (\Phi_0, \ldots, \Phi_{n-1}) : W_n(B) \to W_n(B)$ which is a lift of the Frobenius endomorphism on $W_n(B/pB)$. In particular, set B to be the polynomial ring $\mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$. Put $X = (X_0, \ldots, X_{n-1})$ and $Y = (Y_0, \ldots, Y_{n-1})$ in the ring $W_n(B)$. Then we see that there exists elements $U_0, \ldots, U_{n-1}$ and $U'_0, \ldots, U'_{n-1}$ of the polynomial ring B such that

$$\Phi(X+Y) = \Phi(X) + \Phi(Y) + (pU_0, \ldots, pU_{n-1}),$$
$$\Phi(XY) = \Phi(X)\Phi(Y) + (pU'_0, \ldots, pU'_{n-1})$$

in the ring $W_n(B)$.

Proposition 4.10. *Every d-tuple $(\bar{x}_1, \ldots, \bar{x}_d)$ in $\bar{A}_{n,r+}$ satisfying the above three conditions uniquely lifts to a d-tuple $(\hat{x}_1, \ldots, \hat{x}_d)$ in $W_n(\mathcal{O}_{\bar{K}})$ such that*

- $\hat{s}_{l,1}\hat{x}_1 + \cdots + \hat{s}_{l,d}\hat{x}_d \in ([\zeta_{p^n}] - 1)^r W_n(m_{\bar{K}})$ *for any l,*

- $\hat{c}_{1,i}\hat{x}_1 + \cdots + \hat{c}_{d,i}\hat{x}_d \in \hat{\gamma}^r W_n(\mathcal{O}_{\bar{K}})$ *for any* i,
- *the following equality holds:*

$$\begin{cases} \hat{c}^r\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\gamma}^r) = \hat{x}_1 \\ \vdots \\ \hat{c}^r\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\gamma}^r) = \hat{x}_d. \end{cases}$$

Proof. Fix a lift $\hat{x}_i$ of $\bar{x}_i$ in $W_n(\mathcal{O}_{\bar{K}})$. Recall that the kernel of the surjection $W_n(\mathcal{O}_{\bar{K}}) \to \bar{A}_{n,r+}$ is equal to the ideal $([\zeta_{p^n}]-1)^r W_n(m_{\bar{K}})$. The first condition in the proposition holds automatically for $(\hat{x}_1, \ldots, \hat{x}_d)$. By Lemma 4.7, the element $\hat{c}_{1,i}\hat{x}_1 + \cdots + \hat{c}_{d,i}\hat{x}_d$ is contained in $\hat{\gamma}^r W_n(\mathcal{O}_{\bar{K}})$ for any i. Since the map $\phi_r : \mathrm{Fil}^r \bar{A}_{n,r+} \to \bar{A}_{n,r+}$ satisfies $\phi_r(E([\pi_n])^r\bar{x}) = c^r\phi(\bar{x})$ for any $\bar{x} \in \bar{A}_{n,r+}$, we have

$$\begin{cases} \hat{c}^r\Phi((\hat{c}_{1,1}\hat{x}_1 + \cdots + \hat{c}_{d,1}\hat{x}_d)/\hat{\gamma}^r) = \hat{x}_1 + ([\zeta_{p^n}]-1)^r\hat{\delta}_1 \\ \vdots \\ \hat{c}^r\Phi((\hat{c}_{1,d}\hat{x}_1 + \cdots + \hat{c}_{d,d}\hat{x}_d)/\hat{\gamma}^r) = \hat{x}_d + ([\zeta_{p^n}]-1)^r\hat{\delta}_d \end{cases}$$

for some $\hat{\delta}_1, \ldots, \hat{\delta}_d \in W_n(m_{\bar{K}})$. It suffices to show that there exists a unique d-tuple $(\hat{y}_1, \ldots, \hat{y}_d)$ in $W_n(m_{\bar{K}})$ such that

$$\begin{aligned} \hat{c}^r\Phi((\hat{c}_{1,i}(\hat{x}_1 + ([\zeta_{p^n}]-1)^r\hat{y}_1) + \cdots + \hat{c}_{d,i}(\hat{x}_d + ([\zeta_{p^n}]-1)^r\hat{y}_d))/\hat{\gamma}^r) \\ = \hat{x}_i + ([\zeta_{p^n}]-1)^r\hat{y}_i \end{aligned}$$

for any i. For this, we need the following lemma.

Lemma 4.11. *Let N be a complete discrete valuation field and m_N be the maximal ideal of N. Let $\epsilon_1, \ldots, \epsilon_d$ be in m_N. Let $P_1, \ldots, P_d$ and $P'_1 \ldots, P'_d$ be elements of $\mathcal{O}_N[[Y_1, \ldots, Y_d]]$ such that $P_i \in (Y_1, \ldots, Y_d)^2$. Then the equation*

$$\begin{cases} Y_1 - P_1(Y_1, \ldots, Y_d) - \epsilon_1 P'_1(Y_1, \ldots, Y_d) = 0 \\ \vdots \\ Y_d - P_d(Y_1, \ldots, Y_d) - \epsilon_d P'_d(Y_1, \ldots, Y_d) = 0 \end{cases}$$

has a unique solution in m_N.

Proof. By assumption, we see that for any integer $l \geq 1$, a d-tuple $(y_1, \ldots, y_d)$ in m_N/m_N^l satisfying the above equation lifts uniquely to a d-tuple in m_N/m_N^{l+1} satisfying the same equation. Thus the lemma follows. □

Let us write as $\hat{y}_i = (\hat{y}_{i,0}, \ldots, \hat{y}_{i,n-1})$. Since the image of $\Phi(([\zeta_{p^{n+1}}]-1)^r)$ in $\bar{A}_{n,r+}$ is equal to $([\zeta_{p^n}]-1)^r$, we can find $\hat{b} \in W_n(\mathcal{O}_{\bar{K}})$ such that

$$\Phi(([\zeta_{p^n}]-1)^r/\hat{\gamma}^r) = ([\zeta_{p^n}]-1)^r\hat{b}.$$

Then there exists polynomials $U_{i,m}$ over $\mathcal{O}_{\bar{K}}$ of the indeterminates $\underline{Y} = (Y_{i,m})_{1\leq i\leq d, 0\leq m\leq n-1}$ such that the equation we have to solve is

$$\begin{aligned}\hat{x}_i + ([\zeta_{p^n}]-1)^r\hat{y}_i = \hat{x}_i &+ ([\zeta_{p^n}]-1)^r\hat{\delta}_i \\ &+ ([\zeta_{p^n}]-1)^r\hat{b}\hat{c}^r(\Phi(\hat{c}_{1,i})\Phi(\hat{y}_1)+\cdots+\Phi(\hat{c}_{d,i})\Phi(\hat{y}_d)) \\ &+ (pU_{i,0}(\underline{\hat{y}}),\ldots,pU_{i,n-1}(\underline{\hat{y}}))\end{aligned}$$

for any i, where we put $\underline{\hat{y}} = (\hat{y}_{i,m})_{1\leq i\leq d, 0\leq m\leq n-1}$. As in the proof of Lemma 4.6, we see that, for any elements $P_0,\ldots,P_{n-1}$ of the polynomial ring $\mathcal{O}_{\bar{K}}[\underline{Y}]$, we can uniquely find elements $Q_0,\ldots,Q_{n-1}$ of this ring such that the coefficients of these polynomials are in the maximal ideal $m_{\bar{K}}$ and the equality

$$(pP_0,\ldots,pP_{n-1}) = ([\zeta_{p^n}]-1)^r(Q_0,\ldots,Q_{n-1})$$

holds in the ring of Witt vectors $W_n(\mathcal{O}_{\bar{K}}[\underline{Y}])$. Therefore, this equation is equivalent to the equation

$$\begin{aligned}\hat{y}_i = \hat{\delta}_i &+ \hat{b}\hat{c}^r(\Phi(\hat{c}_{1,i})\Phi(\hat{y}_1)+\cdots+\Phi(\hat{c}_{d,i})\Phi(\hat{y}_d)) \\ &+ (V_{i,0}(\underline{\hat{y}}),\ldots,V_{i,n-1}(\underline{\hat{y}})),\end{aligned}$$

where $V_{i,m}$ is a polynomial of $\underline{Y}$ over $\mathcal{O}_{\bar{K}}$ whose coefficients are in the maximal ideal $m_{\bar{K}}$. From the definition of Φ, we see that $\underline{\hat{y}} = (\hat{y}_{i,m})_{i,m}$ is a solution of a system of equations

$$Y_{i,m} - P_{i,m}(\underline{Y}) - \epsilon_{i,m}P'_{i,m}(\underline{Y}) = 0$$

satisfying the condition of Lemma 4.11 for a sufficiently large finite extension N of K. Then, by this lemma, we can solve the equation uniquely in $m_{\bar{K}}$. □

Let F be an algebraic extension of F_n in $\bar{K}$ and consider the ring $\bar{A}_{n,F,r+}$. By Lemma 4.8, we can consider this ring as a Σ-subalgebra of $\bar{A}_{n,r+}$. Put $\mathrm{Fil}^r\bar{A}_{n,F,r+} = E([\pi_n])^r\bar{A}_{n,F,r+}$. Then Lemma 4.7 implies that

$$\bar{A}_{n,F,r+}\cap \mathrm{Fil}^r\bar{A}_{n,r+} = \mathrm{Fil}^r\bar{A}_{n,F,r+}.$$

Moreover, the Frobenius endomorphism ϕ of the ring $\bar{A}_{n,r+}$ preserves the subalgebra $\bar{A}_{n,F,r+}$ and thus $\phi_r : \mathrm{Fil}^r\bar{A}_{n,r+}\to\bar{A}_{n,r+}$ induces a ϕ-semilinear map $\phi_r : \mathrm{Fil}^r\bar{A}_{n,F,r+}\to\bar{A}_{n,F,r+}$. Hence $\bar{A}_{n,F,r+}$ is a subobject of $\bar{A}_{n,r+}$ in the category $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$. For $M\in\mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$, let us set

$$T^*_{\mathrm{crys},\pi_n,F}(M) = \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M,\bar{A}_{n,F,r+}).$$

We see that

$$\bar{A}_{n,r+} = \bar{A}_{n,\bar{K},r+} = \bigcup_{F/F_n}\bar{A}_{n,F,r+}$$

in $'\mathrm{Mod}^{r,\phi}_{/\Sigma}$ and thus we have a natural identification of abelian groups

$$T^*_{\mathrm{crys},\pi_n,\bar{K}}(M) = \bigcup_{F/F_n} T^*_{\mathrm{crys},\pi_n,F}(M).$$

The absolute Galois group G_{F_n} acts on the abelian group on the left-hand side.

Lemma 4.12. *Let F be an algebraic extension of F_n in $\bar{K}$. Then the G_F-fixed part $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)^{G_F}$ is equal to $T^*_{\mathrm{crys},\pi_n,F}(M)$.*

Proof. From Proposition 4.10, we see that the elements of $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)$ correspond bijectively to the d-tuples in $W_n(\mathcal{O}_{\bar{K}})$ satisfying the three conditions in this proposition. The uniqueness assertion of the proposition shows that $g \in G_F$ fixes such a d-tuple in $W_n(\mathcal{O}_{\bar{K}})$ if and only if g fixes its image in $\bar{A}_{n,r+}$. Hence an element of $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)$ is fixed by G_F if and only if it is contained in the image of $W_n(\mathcal{O}_F)$. Thus the lemma follows. □

Corollary 4.13. *Let L_n be the finite Galois extension of F_n corresponding to the kernel of the map*

$$G_{F_n} \to \mathrm{Aut}(T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)).$$

Then an algebraic extension F of F_n in $\bar{K}$ contains L_n if and only if

$$\# T^*_{\mathrm{crys},\pi_n,F}(M) = \# T^*_{\mathrm{crys},\pi_n,\bar{K}}(M).$$

Proof. An algebraic extension F of F_n contains L_n if and only if the action of G_F on $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)$ is trivial. By Lemma 4.12, this is equivalent to $T^*_{\mathrm{crys},\pi_n,F}(M) = T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)$. □

5. Ramification bound

In this section, we prove Theorem 1.1. Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{r,\phi,N}_{/S_\infty}$ which is killed by p^n and let L be the finite Galois extension of K corresponding to the kernel of the map

$$G_K \to \mathrm{Aut}(T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})).$$

Then the theorem is equivalent to the inequality $u_{L/K} \leq u(K,r,n)$, where $u_{L/K}$ denotes the greatest upper ramification break of the Galois extension L/K ([10]). For $r = 0$, the G_K-module $T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})$ is unramified and the assertion is trivial. Thus we may assume $p \geq 3$ and $r \geq 1$.

Let L_n be the finite Galois extension of F_n corresponding to the kernel of the map

$$G_{F_n} \to \mathrm{Aut}(T^*_{\mathrm{st},\underline{\pi}}(\mathcal{M})).$$

Since F_n is Galois over K, the extension $L_n = LF_n$ is also a Galois extension of K. Let $M \in \mathrm{Mod}^{r,\phi}_{/\Sigma_\infty}$ be the filtered ϕ_r-module over Σ which corresponds to $\mathcal{M}$ by the equivalence $\mathcal{M}_{\Sigma_\infty}$ of Proposition 3.13. Then Proposition 3.15 and Proposition 4.9 show that L_n is also the finite extension of F_n cut out by the G_{F_n}-module $T^*_{\mathrm{crys},\pi_n,\bar{K}}(M)$. It is enough to prove the inequality $u_{L_n/K} \leq u(K,r,n)$.

Before proving this, we state some general lemmas to calculate the ramification bound. Let N be a complete discrete valuation field of positive residue characteristic, v_N be its valuation normalized as $v_N(N^\times) = \mathbb{Z}$ and N^{sep} be its separable closure. We extend v_N to any algebraic closure of N.

Lemma 5.1. *Let $f(T) \in \mathcal{O}_N[T]$ be a separable monic polynomial and $z_1, \ldots, z_d$ be the zeros of f in $\mathcal{O}_{N^{\mathrm{sep}}}$. Suppose that the set $\{v_N(z_k - z_i) \mid k = 1, \ldots, d, k \neq i\}$ is independent of i. Put*

$$s_f = \sum_{\substack{k=1,\ldots,d \\ k\neq i}} v_N(z_k - z_i) \text{ and } \alpha_f = \sup_{\substack{k=1,\ldots,d \\ k\neq i}} v_N(z_k - z_i),$$

which are independent of i by assumption. If $j > s_f + \alpha_f$, then we have the decomposition

$$\{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(f(x)) \geq j\} = \coprod_{i=1,\ldots,d} \{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(x - z_i) \geq j - s_f\}.$$

Otherwise, the set on the left-hand side contains

$$\{x \in \mathcal{O}_{N^{\mathrm{sep}}} \mid v_N(x - z_i) \geq \alpha_f\},$$

which contains at least two zeros of f.

Proof. A verbatim argument in the proof of [1, Lemma 6.6] shows the claim. □

Corollary 5.2. *Let $f(T)$ be as above and put $B = \mathcal{O}_N[T]/(f(T))$. Let us write the N-algebra $N' = B \otimes_{\mathcal{O}_N} N$ as the product $N_1 \times \cdots \times N_t$ of finite separable extensions $N_1, \ldots, N_t$ of N. If $j > s_f + \alpha_f$, then the j-th upper numbering ramification group ([1]), which we let be denoted by $G_N^{(j)}$, is contained in G_{N_i} for any i. Moreover, if N' is a field and B coincides with $\mathcal{O}_{N'}$, then $j > s_f + \alpha_f$ if and only if $G_N^{(j)} \subseteq G_{N'}$.*

Proof. Note that the algebra B is finite flat and of relative complete intersection over $\mathcal{O}_N$. By the previous lemma, the conductor $c(B)$ of the $\mathcal{O}_N$-algebra B ([1, Proposition 6.4]) is equal to $s_f + \alpha_f$. Thus we have the inequality

$$c(\mathcal{O}_{N_1} \times \cdots \times \mathcal{O}_{N_t}) \leq c(B) = s_f + \alpha_f$$

by the definition of the conductor and a functoriality of the functor $\mathcal{F}^j$ defined in [1]. This implies the corollary. □

Corollary 5.3. *We have the inequality*

$$u_{K(\zeta_{p^{n+1}})/K} \leq 1 - \frac{1}{e(K(\zeta_p)/K)} + e(n + \frac{1}{p-1}),$$

where $e(K(\zeta_p)/K)$ denotes the relative ramification index of $K(\zeta_p)$ over K.

Proof. Since the Herbrand function is transitive and the finite extension $K(\zeta_p)$ is tamely ramified over K, it is enough to show the inequality

$$u_{K(\zeta_{p^{n+1}})/K(\zeta_p)} \leq e(K(\zeta_p))(n + \frac{1}{p-1}).$$

Put $N = K(\zeta_p)$ and $f(T) = T^{p^n} - \zeta_p$. These satisfy the assumptions of Corollary 5.2. We have $s_f = ne(K(\zeta_p))$ and $\alpha_f = e(K(\zeta_p))/(p-1)$ in this case. Hence the corollary follows. □

Corollary 5.4. *Consider the finite Galois extension $F_n = K_n(\zeta_{p^{n+1}})$ of K. Then we have the equality*

$$u_{F_n/K} = 1 + e(n + \frac{1}{p-1}).$$

Proof. Applying Corollary 5.2 to the Eisenstein polynomial $f(T) = T^{p^n} - \pi$ and $N = K$ shows that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{K_n}$. From Corollary 5.3, we see that if $j > 1 + e(n + 1/(p-1))$, then $G_K^{(j)} \subseteq G_{K(\zeta_{p^{n+1}})}$. Since $G_{F_n} = G_{K_n} \cap G_{K(\zeta_{p^{n+1}})}$, we conclude that $j > 1 + e(n + 1/(p-1))$ if and only if $G_K^{(j)} \subseteq G_{F_n}$. $\square$

Remark 5.5. Note that this argument also shows the equality

$$u_{K_n(\zeta_{p^n})/K} = 1 + e(n + \frac{1}{p-1}).$$

Next we assume that the residue field of N is perfect. For an algebraic extension F of N, we put

$$\mathfrak{a}^j_{F/N} = \{x \in \mathcal{O}_F \mid v_N(x) \geq j\}.$$

Let Q be a finite Galois extension of N and consider the property

$$(P_j)\begin{cases} \text{for any algebraic extension } F \text{ of } N, \text{ if there exists} \\ \text{an } \mathcal{O}_N\text{-algebra homomorphism } \mathcal{O}_Q \to \mathcal{O}_F/\mathfrak{a}^j_{F/N}, \\ \text{then there exists an } N\text{-algebra injection } Q \to F \end{cases}$$

for $j \in \mathbb{R}_{\geq 0}$, as in [10, Proposition 1.5]. Then we have the following proposition, which is due to Yoshida. Here we reproduce his proof for the convenience of the reader.

Proposition 5.6 ([19])**.**

$$u_{Q/N} = \inf\{j \in \mathbb{R}_{\geq 0} \mid \textit{the property } (P_j) \textit{ holds}\}.$$

Proof. By [10, Proposition 1.5 (i)], it is enough to show that the property (P_j) does not hold for $j = u_{Q/N} - (e')^{-1}$ with an arbitrarily large $e' > 0$. As in the proof of [10, Proposition 1.5 (ii)], we may assume that Q is totally and wildly ramified over N. Take an arbitrarily large integer $e'' > 0$ with $(e'', pe(Q/N)) = 1$. We may also assume that N contains a primitive e''-th root of unity. Set $N' = N(\pi_N^{1/e''})$ and $Q' = QN'$. Note that we have $u_{Q'/N} = u_{Q/N}$ by assumption. From this proposition in [10], we see that for some algebraic extension F of N, there exists an $\mathcal{O}_N$-algebra homomorphism $\mathcal{O}_{Q'} \to \mathcal{O}_F/\mathfrak{a}^j_{F/N}$ for $j = u_{Q/N} - e(Q'/N)^{-1}$ but no N-algebra injection $Q' \to F$. Since Q/N is wildly ramified, we see that $e(Q/N)u_{Q/N} - 1 > e(Q/N)$. Hence we have $u_{Q/N} - e(Q'/N)^{-1} > 1 \geq u_{N'/N}$ and there exists an N-algebra injection $N' \to F$ also by this proposition. Thus there exists no N-algebra injection $Q \to F$ and the property (P_j) for Q/N does not hold. Since $e(Q'/N) = e''e(Q/N)$, the proposition follows. $\square$

We see from Proposition 5.6 that to bound the greatest upper ramification break $u_{L_n/K}$, it is enough to show the following proposition.

Proposition 5.7. *Let F be an algebraic extension of K. If $j > u(K, r, n)$ and there exists an $\mathcal{O}_K$-algebra homomorphism*

$$\eta : \mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{a}^j_{F/K},$$

then there exists a K-algebra injection $L_n \to F$.

Proof. We may assume that F is contained in $\bar{K}$. By assumption, we have $j > er/(p-1)$ and we see that the ideal $\mathfrak{b}_F = \{x \in \mathcal{O}_F \mid v_K(x) > er/(p-1)\}$ contains $\mathfrak{a}^j_{F/K}$. Thus η induces an $\mathcal{O}_K$-algebra homomorphism

$$\mathcal{O}_{L_n} \to \mathcal{O}_F/\mathfrak{b}_F.$$

Since η also induces an $\mathcal{O}_K$-algebra homomorphism $\mathcal{O}_{F_n} \to \mathcal{O}_F/\mathfrak{a}^j_{F/K}$ and $r \geq 1$, from Corollary 5.4 and [10, Proposition 1.5] we get a K-linear injection $F_n \to F$. Thus we see that F contains π_n and $\zeta_{p^{n+1}}$. More precisely, we have the following two lemmas.

Lemma 5.8. *There exists $i \in \mathbb{Z}$ such that $\eta(\pi_n) \equiv \pi_n \zeta^i_{p^n} \bmod \mathfrak{b}_F$.*

Proof. Since the map η is $\mathcal{O}_K$-linear, the equality $\eta(\pi_n)^{p^n} = \pi$ holds in $\mathcal{O}_F/\mathfrak{a}^j_{F/K}$. Set $\hat{x}$ to be a lift of $\eta(\pi_n)$ in $\mathcal{O}_F$. Then we have

$$v_K(\hat{x}^{p^n} - \pi) = \sum_{i=0}^{p^n-1} v_K(\hat{x} - \pi_n \zeta^i_{p^n}) \geq j.$$

Let us apply Lemma 5.1 to $f(T) = T^{p^n} - \pi \in \mathcal{O}_K[T]$. Then, with the notation of the lemma, we have

$$s_f = 1 - \frac{1}{p^n} + ne \text{ and } \alpha_f = \frac{1}{p^n} + \frac{e}{p-1}.$$

Since $j - s_f > er/(p-1)$ by assumption, we have

$$\hat{x} \equiv \pi_n \zeta^i_{p^n} \bmod \mathfrak{b}_F$$

for some i. □

Lemma 5.9. *There exists $g' \in G_K$ such that $\eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \bmod \mathfrak{b}_F$.*

Proof. Set N to be the maximal unramified subextension of $K(\zeta_{p^{n+1}})/K$. Since the map $\mathcal{O}_K \to \mathcal{O}_N$ is etale, there exists a K-algebra injection $g_0 : N \to F$ such that $\eta(x) \equiv g_0(x) \bmod \mathfrak{a}^j_{F/K}$ for any $x \in \mathcal{O}_N$. Let ϖ be a uniformizer of $K(\zeta_{p^{n+1}})$ and $f(T) \in \mathcal{O}_N[T]$ be the Eisenstein polynomial of ϖ over $\mathcal{O}_N$. We let $f^{g_0}(T) \in \mathcal{O}_N[T]$ denote the conjugate of f by g_0. Then f^{g_0} satisfies the conditions of Lemma 5.1. By definition we have $s_{f^{g_0}} = s_f$ and $\alpha_{f^{g_0}} = \alpha_f$. Since the roots of $f^{g_0}(T)$ are conjugates of ϖ over K, Lemma 5.1 implies as in the previous lemma that there exists $g' \in G_K$ such

that $g'|_N = g_0$ and $\eta(\varpi) \equiv g'(\varpi) \bmod \mathfrak{a}_{F/K}^{j-s_f}$. Since $\mathcal{O}_{K(\zeta_{p^{n+1}})}$ is generated by ϖ over $\mathcal{O}_N$, we see that $\eta(\zeta_{p^{n+1}}) \equiv g'(\zeta_{p^{n+1}}) \bmod \mathfrak{a}_{F/K}^{j-s_f}$.

Thus it is enough to check the inequality $j - s_f > er/(p-1)$. Note that s_f is equal to the valuation $v_K(\mathfrak{D}_{K(\zeta_{p^{n+1}})/N})$ of the different of the totally ramified Galois extension $K(\zeta_{p^{n+1}})/N$. To bound this, put $G = \mathrm{Gal}(K(\zeta_{p^{n+1}})/N(\zeta_p))$ and $e' = e(N(\zeta_p)/N)$. We have

$$v_K(\tau(\varpi) - \varpi)) \leq v_K(\tau(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}})$$

for any $\tau \in G$ and thus

$$v_K(\mathfrak{D}_{K(\zeta_{p^{n+1}})/N(\zeta_p)}) \leq \sum_{\tau \neq 1 \in G} v_K(\tau(\zeta_{p^{n+1}}) - \zeta_{p^{n+1}}) \leq ne.$$

We also have the equality $v_K(\mathfrak{D}_{N(\zeta_p)/N}) = 1 - 1/e'$ and hence we get

$$s_f = v_K(\mathfrak{D}_{K(\zeta_{p^{n+1}})/N}) \leq 1 - 1/e' + ne.$$

Since $e' \leq p-1$, the inequality $j - s_f > er/(p-1)$ holds. □

Corollary 5.10. *There exists $g \in G_K$ such that $\eta(\pi_n) \equiv g(\pi_n) \bmod \mathfrak{b}_F$ and $\eta(\zeta_{p^{n+1}}) \equiv g(\zeta_{p^{n+1}}) \bmod \mathfrak{b}_F$.*

Proof. Let $i \in \mathbb{Z}$ and $g' \in G_K$ be as in Lemma 5.8 and Lemma 5.9, respectively. Since $K_n \cap K(\zeta_{p^{n+1}}) = K$ (see for example [17, Lemma 5.1.2]), we can find an element $g \in G_K$ such that $g(\pi_n) = \pi_n \zeta_{p^n}^i$ and $g(\zeta_{p^{n+1}}) = g'(\zeta_{p^{n+1}})$. □

Lemma 5.11. *For $m \in \mathbb{Z}_{\geq 0}$, set an ideal $\mathfrak{b}_{L_n}^{(m)}$ of $\mathcal{O}_{L_n}$ to be*

$$\mathfrak{b}_{L_n}^{(m)} = \{x \in \mathcal{O}_{L_n} \mid v_K(x) > \frac{er}{p^m(p-1)}\}$$

and similarly for F. Then the $\mathcal{O}_K$-algebra homomorphism η induces an $\mathcal{O}_K$-algebra injection

$$\eta^{(m)} : \mathcal{O}_{L_n}/\mathfrak{b}_{L_n}^{(m)} \to \mathcal{O}_F/\mathfrak{b}_F^{(m)}$$

for any m.

Proof. We may assume that L_n is totally ramified over K. We write the Eisenstein polynomial of a uniformizer π_{L_n} of L_n over $\mathcal{O}_K$ as

$$P(T) = T^{e'} + c_1 T^{e'-1} + \cdots + c_{e'-1}T + c_{e'},$$

where $e' = e(L_n/K)$. Then $z = \eta(\pi_{L_n})$ satisfies $P(z) = 0$ in $\mathcal{O}_F/\mathfrak{a}_{F/K}^j$. Let $\hat{z}$ be a lift of z in $\mathcal{O}_F$. Since $j > 1$, we have $v_K(\hat{z}) = 1/e'$. The condition $i > e(L_n)r/(p^m(p-1))$ is equivalent to the condition

$$v_K(\hat{z}^i) > \frac{e(L_n)r}{p^m(p-1)} \cdot \frac{1}{e'} = \frac{er}{p^m(p-1)}.$$

Thus the claim follows. □

Since L_n contains F_n, we can consider the ring

$$\bar{A}_{n,L_n,r+} = W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})/([\zeta_{p^n}]-1)^r W_n(m_{L_n}/\mathfrak{b}_{L_n})$$

and similarly $\bar{A}_{n,F,r+}$ for F. We give these rings structures of Σ-algebras as follows. The ring $\bar{A}_{n,L_n,r+}$ is considered as a Σ-algebra by using the system $\{\pi_n\}_{n\in\mathbb{Z}_{\geq 0}}$ we chose of p-power roots of π, as in the previous section. On the other hand, using $g \in G_K$ in Corollary 5.10, put $\tilde{\pi}_n = g(\pi_n)$ and $\tilde{\zeta}_{p^{n+1}} = g(\zeta_{p^{n+1}})$. Then we consider the ring $\bar{A}_{n,F,r+}$ as a Σ-algebra by using a system of p-power roots of π containing $\tilde{\pi}_n$. We define Fil^r and ϕ_r of these rings in the same way as before.

Lemma 5.12. *The induced ring homomorphism*

$$\bar{\eta} : \bar{A}_{n,L_n,r+} \to \bar{A}_{n,F,r+}$$

is a morphism of the category ${}'\mathrm{Mod}^{r,\phi}_{/\Sigma}$.

Proof. Firstly, we check that $\bar{\eta}$ is Σ-linear. By definition, this homomorphism commutes with the action of the element $u \in \Sigma$. To show the compatibility with the element $Y \in \Sigma$, let us consider the commutative diagram

$$\begin{array}{ccc} W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n}) & \xrightarrow{\eta_n} & W_n(\mathcal{O}_F/\mathfrak{b}_F) \\ \downarrow & & \downarrow \\ \bar{A}_{n,L_n,r+} & \xrightarrow{\bar{\eta}} & \bar{A}_{n,F,r+}, \end{array}$$

where the horizontal arrows are induced by η. Note that we have $\eta_n([\pi_n]) = [\tilde{\pi}_n]$ and $\eta_n([\zeta_{p^{n+1}}]) = [\tilde{\zeta}_{p^{n+1}}]$. Let $a \in W(R)^\times$ and $v = t/E([\underline{\pi}]) \in W(R)^\times$ be as in the previous section. Let a_n and v_n denote the images of a and v in $W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})$, respectively. Then the element v_n is a solution of the equation

$$E([\pi_n])v_n = 1 + [\zeta_{p^{n+1}}] + \cdots + [\zeta_{p^{n+1}}]^{p-1}.$$

Similarly, we define elements $\tilde{a}_n$ and $\tilde{v}_n$ of $W_n(\mathcal{O}_F/\mathfrak{b}_F)$ using $\tilde{\pi}_n$ and $\tilde{\zeta}_{p^{n+1}}$. By definition, the element $\tilde{v}_n$ is a solution of the equation

$$E([\tilde{\pi}_n])\tilde{v}_n = 1 + [\tilde{\zeta}_{p^{n+1}}] + \cdots + [\tilde{\zeta}_{p^{n+1}}]^{p-1}.$$

Now what we have to show is the equality

$$\bar{\eta}(a_n v_n^{-1} E([\pi_n])^{p-1}) = \tilde{a}_n \tilde{v}_n^{-1} E([\tilde{\pi}_n])^{p-1}$$

in the ring $\bar{A}_{n,F,r+}$. Since the element a_n of $W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})$ is a linear combination of the elements $1, [\zeta_{p^{n+1}}], \ldots, [\zeta_{p^{n+1}}]^{p-1}$ over $\mathbb{Z}$, we have $\bar{\eta}(a_n) = \tilde{a}_n$ in $\bar{A}_{n,F,r+}$. The elements $\tilde{v}_n$ and $\bar{\eta}(v_n)$ satisfy the same equation in $\bar{A}_{n,F,r+}$. Since these two elements are invertible, we get $\bar{\eta}(v_n)^{-1}E([\tilde{\pi}_n]) = \tilde{v}_n^{-1}E([\tilde{\pi}_n])$ and the equality holds. Since the diagram above is compatible with the Frobenius endomorphisms, we see from the definition that $\bar{\eta}$ also preserves Fil^r and commutes with ϕ_r of both sides. $\square$

Thus we get a homomorphism of abelian groups

$$T^*_{\mathrm{crys},L_n,\pi_n}(M) \to T^*_{\mathrm{crys},F,\tilde{\pi}_n}(M).$$

Then the following lemma, whose proof is omitted in [3, Subsection 3.13], implies that this homomorphism is an injection. We insert here a proof of this lemma for the convenience of the reader.

Lemma 5.13. *The ring homomorphism $\bar{\eta} : \bar{A}_{n,L_n,r+} \to \bar{A}_{n,F,r+}$ is an injection.*

Proof. Let $x = (x_0, \ldots, x_{n-1})$ be an element of $W_n(\mathcal{O}_{L_n}/\mathfrak{b}_{L_n})$ such that

$$(\eta^{(0)}(x_0), \ldots, \eta^{(0)}(x_{n-1})) \in ([\zeta_{p^n}] - 1)^r W_n(m_F/\mathfrak{b}_F),$$

where $\eta^{(0)}$ is as in Lemma 5.11. Suppose that $x_0 = \cdots = x_{m-1} = 0$ for some $0 \leq m \leq n-1$. Let $\hat{z}_i \in \mathcal{O}_F$ be a lift of $\eta^{(0)}(x_i)$. By Lemma 4.6, we have

$$(0, \ldots, 0, \hat{z}_m, \ldots, \hat{z}_{n-1}) = ([\zeta_{p^n}] - 1)^r (\hat{y}_0, \ldots, \hat{y}_{n-1})$$

for some $\hat{y}_0, \ldots, \hat{y}_{n-1} \in m_F$. Thus we get $\hat{y}_0 = \cdots = \hat{y}_{m-1} = 0$ and $v_K(\hat{z}_m) > er/(p^{n-1-m}(p-1))$. Then Lemma 5.11 implies that x_m is contained in the ideal $\mathfrak{b}_{L_n}^{(n-1-m)}/\mathfrak{b}_{L_n}$ and

$$x = ([\zeta_{p^n}] - 1)^r (0, \ldots, 0, y, 0, \ldots, 0) + (0, \ldots, 0, x'_{m+1}, \ldots, x'_{n-1})$$

for some $y \in m_{L_n}/\mathfrak{b}_{L_n}$ and $x'_{m+1}, \ldots, x'_{n-1} \in \mathcal{O}_{L_n}/\mathfrak{b}_{L_n}$. Repeating this, we see that x is zero in $\bar{A}_{n,L_n,r+}$ and the lemma follows. $\square$

Now Corollary 4.13 shows that the abelian group $T^*_{\mathrm{crys},L_n,\pi_n}(M)$ has the same cardinality as $T^*_{\mathrm{crys},\bar{K},\pi_n}(M)$. This implies that the abelian group $T^*_{\mathrm{crys},F,\tilde{\pi}_n}(M)$ has cardinality no less than $\# T^*_{\mathrm{crys},\bar{K},\pi_n}(M)$. Let $g \in G_K$ be as in Corollary 5.10. Then we have the following lemma.

Lemma 5.14. *The G_{F_n}-module $T^*_{\mathrm{crys},\bar{K},\tilde{\pi}_n}(M)$ is isomorphic to the conjugate of the G_{F_n}-module $T^*_{\mathrm{crys},\bar{K},\pi_n}(M)$ by the element g.*

Proof. Let us consider the composite

$$\Sigma \to \bar{A}_{n,r+} \xrightarrow{g} \bar{A}_{n,r+}$$

of the ring homomorphism defined by $u \mapsto [\pi_n]$ and the map induced by g. We can check that this is the natural ring homomorphism defined by $u \mapsto [\tilde{\pi}_n]$ as in the proof of Lemma 5.12. Thus we have an isomorphism of abelian groups

$$\begin{aligned} \mathrm{Hom}_\Sigma(M, \bar{A}_{n,r+}) &\to \mathrm{Hom}_\Sigma(M, \bar{A}_{n,r+}) \\ f &\mapsto g \circ f, \end{aligned}$$

where we consider on the ring $\bar{A}_{n,r+}$ on the right-hand side the filtered ϕ_r-module structure over Σ defined by $\tilde{\pi}_n$. As in the proof of Lemma 5.12, we can check that this isomorphism induces an injection

$$\mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \bar{A}_{n,r+}) \to \mathrm{Hom}_{\Sigma,\mathrm{Fil}^r,\phi_r}(M, \bar{A}_{n,r+}).$$

This is also an isomorphism, for the map $f \mapsto g^{-1} \circ f$ defines its inverse. $\square$

Thus we have $\# T^*_{\mathrm{crys},\bar{K},\tilde{\pi}_n}(M) = \# T^*_{\mathrm{crys},\bar{K},\pi_n}(M)$. Since L_n is Galois over K, this lemma also shows that the finite Galois extension of F_n cut out by the action on $T^*_{\mathrm{crys},\bar{K},\tilde{\pi}_n}(M)$ is L_n. Hence we see from Corollary 4.13 that F contains L_n and Proposition 5.7 follows. This concludes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.3. The second assertion follows immediately from Theorem 1.1 and [8, Théorème 1.1]. As for the first assertion, note that if $r = 0$ then V is unramified and the assertion is trivial. Thus we may assume $p \geq 3$. Since we have the natural surjection $\mathcal{L}/p^n\mathcal{L} \to \mathcal{L}/\mathcal{L}'$, we may also assume $\mathcal{L}' = p^n\mathcal{L}$. For $\hat{\mathcal{M}} \in \mathrm{Mod}^{r,\phi,N}_{/S}$, let us consider the G_K-module

$$T^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\hat{\mathcal{M}}, \hat{A}_{\mathrm{st}}).$$

By [17, Theorem 2.3.5], there exists $\hat{\mathcal{M}} \in \mathrm{Mod}^{r,\phi,N}_{/S}$ such that the G_K-module $\mathcal{L}$ is isomorphic to $T^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}})$. Then we see that the G_K-module $\mathcal{L}/p^n\mathcal{L}$ is isomorphic to $T^*_{\mathrm{st},\underline{\pi}}(\hat{\mathcal{M}}/p^n\hat{\mathcal{M}})$ and the assertion follows from Theorem 1.1. $\square$

Remark 5.15. The ramification bound in Theorem 1.1 is sharp for $r \leq 1$. Indeed, the greatest upper ramification break $1 + e(n + 1/(p-1))$ for $r = 1$ is obtained by the p^n-torsion of the Tate curve $\bar{K}^\times/\pi^{\mathbb{Z}}$ (see Remark 5.5). The author does not know whether these bounds are sharp also for $r \geq 2$.

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